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HANKEL FUNCTION IN MATHEMATICS

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ABSTRACT

The Hankel function is a complex-valued solution to Bessel's differential equation. These functions are very useful for problems involving spherical and cylindrical wave propagation.

It is also called a Bessel function of the third kind, or a Weber Function.

In essence, a Hankel Function is a combination of Bessel functions of the first kind and second kind. So you can think of it as a "type" of Bessel function. In many mathematical programs, the Hankel is defined in terms of the Bessel. For example, in MATLAB, the Hankel function syntax is:

- First Kind: H = besselh(nu,Z),
- First or Second Kind: H = besselh(nu,K,Z).

Where:

- k = each element (1 or 2) of the complex array z,
- nu = the order of the Hankel function. This must be the same size as Z, or one can be scalar. For example, besselh(4,Z).

As far as evaluating the function, you're probably going to want to use software, because it is notoriously difficult to evaluate numerically by hand, especially for large order and large argument (Jentschura, 2011).

KEYWORDS-Hankel,Function,Weber,First,Second,Kind,Software,Argument

I. INTRODUCTION

Hankel Function of the First Kind

The Hankel function of the first kind is defined as:

$$H_n^{(1)}(z) \equiv J_n(z) + iY_n(z)$$

Where:

- $J_{n(z)}$ = Bessel function of the first kind,

- $Y_n(z)$ = Bessel function of the second kind.[1,2,3]

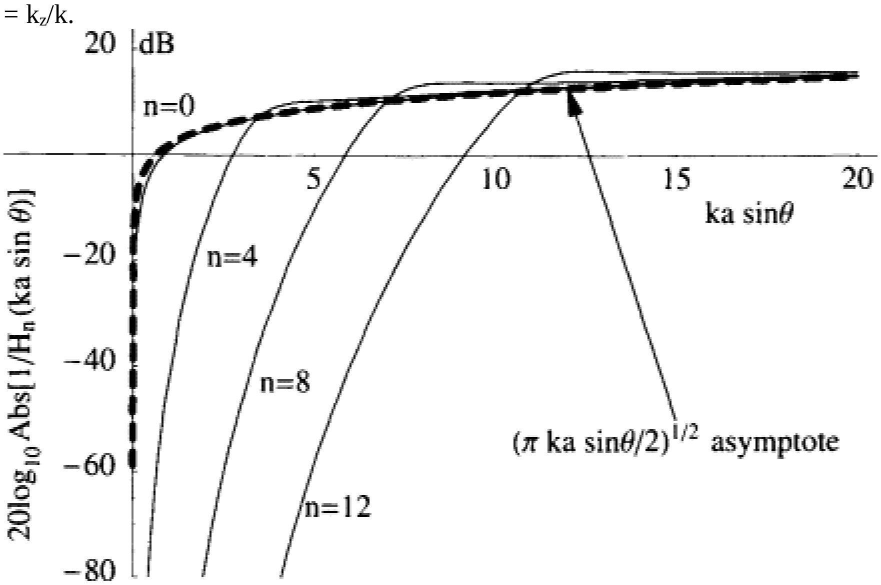
- Contour Integral Definition-The Hankel function can also be represented by the following contour integral:

$$H_n^{(1)}(z) = \frac{1}{i\pi} \int_{0\ \text{[upper half plane]}}^{\infty} \frac{e^{(z/2)(t - 1/t)}}{t^{\,n+1}}\, dt.$$

There are two types of functions known as Hankel functions. The more common one is a complex function (also called a Bessel function of the third kind, or Weber Function) which is a linear combination of Bessel functions of the first and second kinds.

Another type of Hankel function is defined by the contour integral

We see that the Hankel function is a strong filter with respect to n and that when n > ka sin θ the radiation to the farfield is greatly diminished since 1/′ in Eq. becomes very small. Given the break point at ka sin θ = n, this equation forms a circle in k-space for all possible polar angles, 0 ≤ θ ≤ π, and positive and negative n for fixed k. This is shown in Fig. here. The farfield spherical angle θ is the angle between k and the k_z axis and is defined by cos θ = k_z/k.



II. DISCUSSION

The Hankel transform is an integral transform and was first developed by the mathematician Hermann Hankel. It is also known as the Fourier–Bessel transform. Just as the Fourier transform for an infinite interval is related to the Fourier series over a finite interval, so the Hankel transform over an infinite interval is related to the Fourier–Bessel series over a finite interval. In mathematics, the Hankel transform expresses any given function f(r) as the

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weighted sum of an infinite number of Bessel functions of the first kind $J_\nu(kr)$. The Bessel functions in the sum are all of the same order ν, but differ in a scaling factor k along the r axis. The necessary coefficient F_ν of each Bessel function in the sum, as a function of the scaling factor k constitutes the transformed function.

Let $f(r)$ be a function defined for r ≥ 0. The νth order Hankel transform of $f(r)$ is defined as

$$F_\nu(s) \equiv \mathcal{H}_\nu\{f(r)\} \equiv \int_0^\infty r f(r) J_\nu(sr)dr \ .$$

If ν > –1/2, Hankel's repeated integral immediately gives the inversion formula

$$f(r) = \mathcal{H}_\nu^{-1}\{F_\nu(s)\} \equiv \int_0^\infty sF_\nu(s) J_\nu(sr)ds \ .$$

The most important special cases of the Hankel transform correspond to ν = 0 and ν = 1. Sufficient but not necessary conditions for the validity[5,7,8]

1. f (r) = O(r –k), r → ∞ where k > 3/2.
2. f ′(r) is piecewise continuous over each bounded subinterval of [0, ∞).
3. f(r) is defined as [f(r+) + f(r–)]/2.

Connection with the Fourier Transform

We consider the two-dimensional Fourier transform of a function ϕ(x,y), which shows a circular symmetry. This means that ϕ(r cos θ, r sin θ) $f(r,\theta)$ is independent of θ. The Fourier transform of ϕ is

$$\Phi(\zeta,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j(x\zeta,y\eta)}dx\,dy \ .$$

We introduce the polar coordinates x = rcosθ, y = rsin θ, and ζ = scosϕ, η = ssinϕ.

We have then

$$\phi(s\cos\varphi, s\sin\varphi) \equiv F(s,\varphi) = \frac{1}{2\pi}\int_0^\infty r\,dr\int_0^{2\pi} e^{-jrs\cos(\theta-\varphi)} f(r)d\theta$$

$$= \frac{1}{2\pi}\int_0^\infty rf(r)dr\int_0^{2\pi} e^{-jrs\cos\alpha}d\alpha$$

$$= \int_0^\infty rf(r)J_0(rs)dr.$$

This result shows that F(s,ϕ) is independent of ϕ, so that we can write F(s) instead of F(s, ϕ). Thus, the two-dimensional Fourier transform of a circularly symmetric function is, in fact, a Hankel transform of order zero. This result can be generalized: the N-dimensional Fourier transform of a circularly symmetric function of N variables is related to the Hankel transform of order N/2 – 1. If f(r,θ) depends on θ, we can expand it into a Fourier series

$$f(r,\theta)=\sum_{n=-\infty}^{\infty} f_n(r)e^{jn\theta}$$

and, similarly

$$F(s,\varphi)=\frac{1}{2\pi}\int_0^{\infty} r\,dr\int_0^{\infty} e^{-jrs\cos(\theta-\varphi)}f(r,\theta)d\theta=\sum_{n=-\infty}^{\infty} F_n(s)e^{jn\varphi}$$

where

$$f_n(r)=\frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)e^{-jn\theta}d\theta$$

and

$$F_n(s)=\frac{1}{2\pi}\int_0^{2\pi} F(s,\varphi)e^{-jn\varphi}d\varphi \quad .$$

$$F_n(s)=\frac{1}{(2\pi)^2}\int_0^{2\pi} e^{-jn\varphi}d\varphi\int_0^{2\pi} d\theta\int_0^{\infty} f(r,\theta)e^{jsr\cos(\theta-\varphi)}r\,dr$$

$$=\frac{1}{(2\pi)^2}\int_0^{2\pi} e^{-jn\varphi}d\varphi\int_0^{\infty} r\,dr\int_0^{2\pi} e^{jsr\cos(\theta-\varphi)}d\theta\times\sum_{m=-\infty}^{\infty} f_m(r)e^{jm\theta}$$

$$=\frac{1}{(2\pi)}\int_0^{\infty} r\,dr\int_0^{2\pi} e^{-jn\alpha}e^{jsr\cos\alpha}f_n(r)d\alpha$$

$$=\int_0^{\infty} rf_n(r)J_n(sr)dr$$

$$=\mathcal{H}_n\{f_n(r)\}.$$

In a similar way, we can derive

$$f_n(r)=\mathcal{H}_n\{F_n(s)\} \quad .$$

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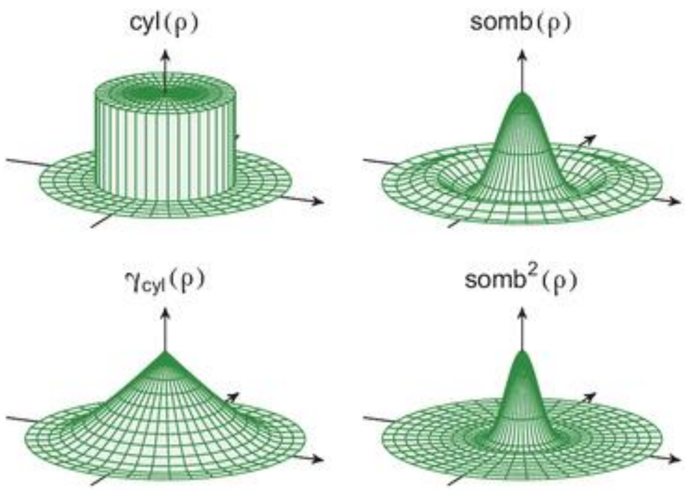
III. RESULTS



Hankel transforms do not have as many elementary properties as do the Laplace or the Fourier transforms. For example, because there is no simple addition formula for Bessel functions, the Hankel transform does not satisfy any simple convolution relation.

1. Derivatives. Let

$$F_\nu(s) = \mathcal{H}_\nu\{f(x)\}\ .$$ Then

Proof,

$$G_\nu(s) = \int_0^\infty x f'(x)\, J_\nu(sx)\, dx$$

$$= [x f(x)\, J_\nu(sx)]_0^\infty - \int_0^\infty f(x) \frac{d}{dx}[x J_\nu(sx)]\, dx\ \cdot$$

In general, the expression between the brackets is zero, and

$$\frac{d}{dx}[x J_\nu(sx)] = \frac{sx}{2\nu}[(\nu+1)\, J_{\nu-1}(sx) - (\nu-1)\, J_{\nu+1}(sx)]\ .$$

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2. The Hankel transform of the Bessel differential operator. The Bessel differential operator

$$\Delta_\nu \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{\nu}{r}\right)^2 = \frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \left(\frac{\nu}{r}\right)^2$$

is derived from the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

after separation of variables in cylindrical coordinates (r, θ, z).

Let f(r) be an arbitrary function with the property that f(r→∞) = 0. Then

$$\mathcal{H}_\nu\{\Delta_\nu f(r)\} = -s^2 \mathcal{H}_\nu\{f(r)\} \quad .$$

This result shows that the Hankel transform may be a useful tool in solving problems with cylindrical symmetry and involving the Laplacian operator.[9,10,11]

IV. CONCLUSIONS

Hermann Hankel's father was Wilhelm Gottlieb Hankel who was a physicist at Halle at the time Hermann was born. Hermann Hankel was a German mathematician who worked on the theory of complex numbers, the theory of functions and the history of mathematics. He is remembered for the Hankel transform and the Hankel matrix. Hermann began his education in Halle but, in 1849 Wilhelm was appointed to the chair of physics at Leipzig so the family moved to Leipzig where Hermann attended the Nicolai Gymnasium. In 1857 Hankel entered the University of Leipzig where he studied mathematics with Möbius and physics with his own father. Following the tradition in Germany at that time Hankel did not complete his studies at one university, but moved to several different universities during the course of his studies. From Leipzig he went to Göttingen in 1860 where he became a student of Riemann and then, in the following year, he worked with Weierstrass and Kronecker in Berlin. He received his doctorate for a thesis Über eine besondere Classe der symmetrischen Determinanten Ⓣ in 1862.

Hankel's habilitation was accepted in 1863 and he began teaching at Leipzig where he was appointed extraordinary professor in 1867. The appointment as extraordinary professor had been in the spring but by the autumn of the same year Hankel was at Erlangen to take up an appointment as ordinary professor. He married Marie Dippe in Erlangen but again he would move fairly soon, accepting the chair at Tübingen in 1869.

He worked on the theory of complex numbers, the theory of functions and the history of mathematics. His work on complex analysis, however, is not considered of the first rank and in [8] he is included with those who contributed but whose:-

... influence on the foundations of complex analysis was not as essential as that of those mathematicians discussed in more detail [Riemann, Weierstrass, Hurwitz, Bieberbach ...]

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Hankel made a systematic study of the rules of arithmetic with his Prinzip der Permanenz der formalen Gesetze Ⓣ (1867), see [7]. He wrote another important work which was also published in 1867 Theorie der complexen Zahlensysteme Ⓣ which did much to make Grassmann's ideas better known. This work [1]:-

... constitutes a lengthy presentation of much of what was then known of the real, complex, and hypercomplex number systems. Beginning with a revised statement of George Peacock's principle of permanence of formal laws, he developed complex numbers as well as such higher algebraic systems as Möbius's barycentric calculus, some of Hermann Grassmann's algebras, and W R Hamilton's quaternions. Hankel was the first to recognise the significance of Grassmann's long-neglected writings ...

Hankel looked at Riemann's integration theory and restated it in terms of measure theoretic concepts. This, and other work he did in this area, constitutes progress towards our current integration theories. He is remembered for the Hankel transformation which occurs in the study of functions which depend only on the distance from the origin. He also studied functions, now named Hankel functions or Bessel functions of the third kind, in a series of papers which appeared in Mathematische Annalen.

His historical writings are rather hard to evaluate since they contain many errors, yet they are filled with brilliant insight. In the same way that he saw the importance of Grassmann's work, Hankel also must have considerable credit for seeing the importance of Bolzano's work on infinite series.[12]

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