ISSN(Online): 2320-9801
ISSN (Print): 2320-9798

International Journal of Innovative Research in Computer and Communication Engineering

(An ISO 3297: 2007 Certified Organization)

Vol. 3, Issue 5, May 2015

Radio number of square of hypercube

Laxman Saha
Assistant Professor, Department of Mathematics, Balurghat College, Balurghat, India

ABSTRACT: Let Gbe a graph with diameter d. A radio labelling of G is a function f that assigns to each vertex with a non-negative integer such that the following holds for all vertices u,v: $|f(u)-f(v)|\geq d+1-d(u,v)$, where $d(u,v)$ is the distance between u and v. The span of f is the absolute difference of the largest and smallest values in $f(V)$. The radio number of G is the minimum span of a radio labelling admitted by G. In this article we determine the radio number of square of an odd dimensional hypercube.

KEYWORDS: Code, Resolving set, Metric dimension.

I. INTRODUCTION

The Frequency Assignment Problem (FAP) is to assign frequencies to the transmitters in a network in a way which avoids interference and uses the spectrum as efficiently as possible. Sometimes these assigning frequencies are called channels. Thus the problem is also known as the *channel assignment problem*. Hale [5] formalized the frequency assignment problem as a generalized graph coloring problem. This coloring have several variations depending upon the type of assignment of frequencies to stations. If the channels assigned to the stations u and v are $f(u)$ and $f(v)$, respectively, then $|f(u)-f(v)|\geq \ell_{uv}$, where ℓ_{uv} is inversely proportional to the distance $d(u,v)$ between the stations u and v. Chartrand et al.[2] have introduced the radio k-coloring of simple connected graphs by taking $\ell_{uv}=diam(G)+1-d(u,v)$. The *span* of a radio labelling f, denoted by $span_f\ (G)$, is the largest integer assigned to a vertex of G. The *radio number of* G, denoted by $rn(G)$, is the minimum of spans of all possible radio labelings of G. A radio labeling $G$of $G$is called optimal if $span_f(G)=rn(G)$.

Determining the radio chromatic number of a graph is an interesting yet difficult combinatorial problem with potential applications to CAP. The radio number of any hypercube was determined in [7] by using generalized binary Gray codes. Ortiz et al. [20] have studied the radio number of generalized prism graphs and have computed the exact value of radio number for some specific types of generalized prism graphs. For two positive integers $m\geq 3$ and $n\geq 3$, the Toroidal grids $T_{m,n}$ are the cartesian product of cycle C_m with cycle C_n. Morris et al. [19] have determined the radio number of $T_{n.n}$ and Saha et al. [21] have given exact value for radio number of $T_{m,n}$ when $mn\ \equiv\ 0\ (mod\ 2)$. The radio numbers of the square of paths and cycles were studied in [14, 15]. For a cycle C_n, the radio number was determined by Liu and Zhu [16], and the antipodal number is known only for $n\ =\ 1,2,3\ \ (mod\ 4)$ (see [1, 6]).

The square of a graph G is the graph G^2 having the vertex set same as that of G and edges between pair of vertices at distance one or two in G. In this article, we determine the radio number of square of an odd dimensional hypercube.

II. PRELIMINARIES

For a hypercube Q_n of dimension n, the vertex set can be taken as binary n-bit strings and two vertices being adjacent if the corresponding strings differ at exactly one bit. For any two n-bit binary strings $a=a_0a_1\dots a_{n-1}$ and $b=b_0b_1\dots b_{n-1}$ the Hamming distance $d_H(a,b)$ between a and b is the number of bits in which they differ. In particular, if $x,y\in\{0,1\}$, then $d_H(x,y)\ =\ 0$ or 1 according as $x\ =\ y$ or $x\ \neq\ y$. If u and v are two vertices of Q_n with a and b as the corresponding strings, then $d_{Q_n}(u,v)=d_H(a,b)$. Two n-bit binary strings may differ in at most n positions, so diameter of Q_n is n. The results in the following lemma may be found in [17].

Lemma 2.1 *For any three vertices u,v and w in Q_n, the following are hold*

(*a*) $d_{Q_n}(u,v)+\ d_{Q_n}(v,w)+\ d_{Q_n}(w,u)\leq 2n$

(*b*) $d_{Q_n}(u,v)+\ d_{Q_n}(v,w)+\ d_{Q_n}(w,u)=2n$ *if and only if one of* $d_{Q_n}(u,v)$, $d_{Q_n}(v,w)$, $d_{Q_n}(w,u)$ *is* n.

ISSN(Online): 2320-9801
ISSN (Print): 2320-9798

International Journal of Innovative Research in Computer and Communication Engineering

(An ISO 3297: 2007 Certified Organization)

Vol. 3, Issue 5, May 2015

III. LOWER BOUND FOR RADIO NUMBER OF Q_n^2

In this section we give a lower bound for radio number of Q_n^2. In next section we shall show that this bound is sharp when $n \equiv 1, 3 \ (mod\ 4)$. For lower bound we need the following lemma.

Lemma 3.1 For every three vertices u, v and w of Q^2, $d(u,v) + d(v,w) + d(w,u) \leq n + 1$.

Proof : It is easy to see that $d(u,v) = \left\lceil \frac{d_{Q_n}(u,v)}{2} \right\rceil$ for any two vertices u and u of Q_n^2, where $d_{Q_n}(u,v)$ denotes the distance of u and v in Q_n. Therefore, $d_{Q_n}(u,v) = 2d(u,v) - r_1$ where $r_1 \in \{0,1\}$. Similarly, $d_{Q_n}(v,w) = 2d(v,w) - r_2$ and $d_{Q_n}(w,u) = 2d(w,u) - r_3$, where $r_2, r_3 \in \{0,1\}$. Again applying Lemma 2.1, we have $d_{Q_n}(u,v) + d_{Q_n}(v,w) + d_{Q_n}(w,u) \leq 2n$ and so $d(u,v) + d(v,w) + d(w,u) = \frac{1}{2}\{ d_{Q_n}(u,v) + d_{Q_n}(v,w) + d_{Q_n}(w,u) + r_1 + r_2 + r_3\} \leq \frac{2n+r_1+r_2+r_3}{2}$. Since $d(u,v) + d(v,w) + d(w,u)$ is an integer, we get

$$\begin{aligned} d(u,v) + d(v,w) + d(w,u) &\leq \left\lfloor \frac{n + r_1 + r_2 + r_3}{2} \right\rfloor \\ &\leq \left\lfloor \frac{2n+3}{2} \right\rfloor, \qquad (since\ 0 \leq r_1, r_2, r_3 \leq 1). \\ &= n + 1. \end{aligned}$$

Hence the lemma is proved.

Theorem 1 For an n -dimensional hypercube Q_n,

$$rn(Q_n^2) \geq \begin{cases} \left\lceil \frac{n+7}{4} \right\rceil, & if\ n\ is\ odd; \\ \left\lceil \frac{n+4}{4} \right\rceil (2^{n-1} - 1) + 1, & if\ n\ is\ even. \end{cases}$$

Proof : Let f be an arbitrary radio labelling of Q_n^2 and $x_0, x_1, \ldots, x_{2^{n-1}}$ be an ordering of the vertices of Q_n^2 such that $0 = f(x_0) < f(x_1) < \cdots < f(x_n)$. Let D be the diameter of Q_n^2. Then $D = \frac{n}{2}$ or $\frac{n+1}{2}$ according as n is even or odd. Now from the radio conditions, we have

$$\begin{aligned} f(x_{i+1}) - f(x_i) &\geq D + 1 - d(x_i, x_{i+1}) \\ f(x_{i+2}) - f(x_{i+1}) &\geq D + 1 - d(x_{i+1}, x_{i+2}) \\ f(x_{i+2}) - f(x_i) &\geq D + 1 - d(x_i, x_{i+2}). \end{aligned}$$

Now adding the above inequality and using Lemma 3.1, we get

$$2(f(x_{i+1}) - f(x_i)) \geq 3(D+1) - (n+1)$$

and the above inequality holds for each $i \in \{0, 1, \ldots, 2^n - 3\}$. Since $f(x_{i+1}) - f(x_i)$ is an integer, the above inequality gives

$$f(x_{i+2}) - f(x_i) \geq \left\lceil \frac{3(D+1) - (n+1)}{2} \right\rceil, \quad 0 \leq i \leq 2^n - 3.$$

Adding the above inequality for even $i$and using $f(x_{2^n-1}) - f(x_{2^n-2}) \geq 1$, we have

$$f(x_{2^n-1}) \geq \left\lceil \frac{3(D+1) - (n+1)}{2} \right\rceil (2^{n-1} - 1) + 1.$$

Hence the result as f was an arbitrary radio labelling of Q_n^2.

ISSN(Online): 2320-9801
ISSN (Print): 2320-9798

International Journal of Innovative Research in Computer and Communication Engineering

(An ISO 3297: 2007 Certified Organization)

Vol. 3, Issue 5, May 2015

IV. OPTIMAL RADIO LABELLING OF Q_n^2

In this section, we present an optimal radio labelling of Q_n^2 when n is odd.

Lemma 4.1 *For any odd integer* n *, there exists an ordering* $v_0, v_2, \dots, v_{2^n-1}$ *of vertices of* Q_n *such that the sequence* $d(v_0, v_1), d(v_1, v_2), \dots, d(v_{2^n-2}, v_{2^n-1})$ *is an alternating sequence of* n*'s and* $\frac{n+1}{2}$ *(beginning with* n *).*

The lemmas given below produces an ordering of vertices Q_n^2 that facilitate an optimal radio labelling.

Lemma 4.2 *For* $n \equiv 3 \pmod 4$, *there exists an ordering* $v_0, v_2, \dots, v_{2^n-1}$ *of vertices of* Q_n^2 *such that for all* i *the following hold.*

(a) $d(v_i, v_{i+1}) = \begin{cases} \frac{n+1}{2}, & \text{if } i \text{ is even} \\ \frac{n+1}{4}, & \text{if } i \text{ is odd.} \end{cases}$

(b) $d(v_i, v_{i+1}) = \frac{n+1}{4}$.

(c) $d(v_i, v_{i+3}) = \frac{n+1}{4}$ *for even integer* i.

Proof : Since $V(Q) = V(Q_n^2)$, we take the same ordering $v_0, v_2, \dots, v_{2^n-1}$ as in Lemma4.1 for the vertices of Q_n^2. Since $d(v_0, v_1)$, $d(v_1, v_2), \dots$, $d(v_{2^n-2}, v_{2^n-1})$ is an alternating sequence of n's and $\frac{n+1}{2}$ (beginning with n), so the part (a) is complete due to the fact $d(u, v) = \left\lceil \frac{d_{Q_n}(u,v)}{2} \right\rceil$ for any two vertices u, v in Q_n^2. For part (b), first we assume i being an even integer. Then

$$d_{Q_n}(v_i, v_{i+1}) = n$$

and

$$d_{Q_n}(v_{i+1}, v_{i+2}) = \frac{n+1}{2}$$

and hence from Lemma2.1 we have

$$d_{Q_n}(v_{i+2}, v_i) = \frac{n-1}{2}.$$

Therefore,

$$d(v_i, v_{i+1}) = \frac{n+1}{2},$$
$$d(v_{i+1}, v_{i+2}) = \frac{n+1}{4}$$

and

$$d(v_i, v_{i+2}) = \frac{n+1}{4}.$$

By similar argument we can prove the result for odd integer i.

Part (c) is similar to the part of (b). For part (c) we have to take three vertices v_i, v_{i+2}, v_{i+3} for even i.

Using similar argument as used in Lemma 4.2 we can prove the following result.

Lemma 4.3 For $n \equiv 1 \pmod 4$, there exists an ordering $v_0, v_2, \dots, v_{2^n-1}$ of vertices of Q_n^2 such that for all Q_n^2 the following hold.

ISSN(Online): 2320-9801
ISSN (Print): 2320-9798

International Journal of Innovative Research in Computer and Communication Engineering

(An ISO 3297: 2007 Certified Organization)

Vol. 3, Issue 5, May 2015

(a) $d(v_i, v_{i+1}) = \begin{cases} \frac{n+1}{2}, & if\ \ i\ is\ even \\ \frac{n+3}{4}, & if\ i\ is\ odd. \end{cases}$

(b) $d(v_i, v_{i+1}) = \frac{n-1}{4}$.

(c) $d(v_i, v_{i+3}) = \frac{n-1}{4}, \qquad for\ even\ integer\ i.$

In the theorem below we determine an upper bound for radio number of Q_n^2 when n is odd.

Theorem 2 For any n-dimensional hypercube Q_n with n as odd

$$rn(Q_n^2) \leq \begin{cases} \left(\frac{n+7}{4}\right)(2^{n-1} - 1) + 1, & if\ n\ \equiv\ 1\ (mod\ 4); \\ \left(\frac{n+7}{4}\right)(2^{n-1} - 1) + 1, & if\ n\ \equiv\ 1\ (mod\ 4). \end{cases}$$

Proof : Let $v_0, v_2, \ldots, v_{2^n-1}$ be the ordering of vertices of Q_n prescribed in Lemma 4.1. Then $v_0, v_2, \ldots, v_{2^n-1}$ is an arrangement of vertices in Q_n^2 such that the sequence $d(v_0, v_1), d(v_1, v_2), \ldots, d(v_{2^n-2}, v_{2^n-1})$ is an alternating sequence of $\frac{n+1}{2}$'s and $\left\lceil\frac{n+1}{2}\right\rceil$'s $\left(beginning\ with\ \frac{n+1}{2}\right)$. We consider the following two cases according $n \equiv 3\ (mod\ 4)$ and $n\ \equiv\ 1\ (mod\ 4)$. In both the cases we take $V(Q_n^2) = \{\ v_0, v_2, \ldots, v_{2^n-1}\}$.

Case-1 : $n\ \equiv\ 3\ (mod\ 4)$. Here define a mapping $f: V(Q_n^2) \rightarrow \{0,1,2, \ldots.,\}$ by

$$f(v_{i+1}) = \begin{cases} \frac{i}{2}\left(\frac{n+1}{4} + 2\right), & if\ i\ is\ even \\ \frac{i-1}{2}\left(\frac{n+1}{4} + 2\right) + 1, & if\ i\ is\ odd. \end{cases}$$

We show that f is a radio labelling of Q_n^2. Let v_i and v_j be arbitrary two vertices of Q_n^2. Without loss of generality we assume $i < j$. If $j = i + 1$, then from the definition of f,

$$f(v_{i+1}) - f(v_i) = \begin{cases} 1, & if\ i\ is\ even \\ \frac{i+1}{2}\left(\frac{n+1}{4} + 2\right) - \frac{i-1}{2}\left(\frac{n+1}{4} + 2\right) - 1, & if\ i\ is\ odd. \end{cases}$$

$$= \begin{cases} 1, & if\ i\ is\ even \\ \frac{n+1}{4} + 1, & if\ i\ is\ odd. \end{cases}$$

As $d(v_i, v_{i+1}) = \frac{n+1}{2}$ or $d(v_i, v_{i+1}) = \frac{n+1}{4}$ according as i is even or odd, so from the above equality, we may write $f(v_{i+1}) - f(v_i) = \frac{n+1}{2} + 1 - d(v_{i+1}, v_i)$ for all i. Therefore the radio condition is satisfied for $j = i + 1$. Now we show that the same is true for $j \geq i + 4$. For this we calculate $f(v_{i+4}) - f(v_i)$ in the following. From the definition of f we have

ISSN(Online): 2320-9801
ISSN (Print): 2320-9798

International Journal of Innovative Research in Computer and Communication Engineering

(An ISO 3297: 2007 Certified Organization)

Vol. 3, Issue 5, May 2015

$$f(v_{i+4})-f(v_i)=\begin{cases}\frac{i+4}{2}\left(\frac{n+1}{4}+2\right)-\frac{i}{2}\left(\frac{n+1}{4}+2\right), & if\ i\ is\ even\\ \frac{i+3}{2}\left(\frac{n+1}{4}+2\right)+1-\frac{i-1}{2}\left(\frac{n+1}{4}+2\right)-1, & if\ i\ is\ odd\end{cases}$$

$$=\frac{n+1}{2}+4.$$

Therefore $f(v_{i+1})-f(v_i)>\frac{n+1}{2}$ for $j\geq i+4$ as f is an increasing function with j. Since the diameter of Q_n^2 is $\frac{n+1}{2}$ the radio condition is satisfies for two vertices v_i and v_j when $j\geq i+4$. Therefore, our remaining cases for showing radio condition are $j=i+2$ and $j=i+3$. First take $j=i+2$. From the definition of f and using $d(v_i,v_{i+2})=\frac{n+1}{4}$ from Lemma 2.1, we have

$$f(v_{i+2})-f(v_i)=\frac{n+1}{4}+2>\frac{n+1}{2}+1-d(v_i,v_{i+2}) \qquad (1)$$

Therefore radio conditions are also satisfies when $j=i+2$. Now we take $j=i+3$. From definition of f , we have

$$f(v_{i+3})-f(v_i)=\begin{cases}\frac{n+1}{4}+3, & if\ i\ is\ even;\\ \frac{n+1}{2}+3, & if\ i\ is\ odd.\end{cases}$$

From above it is clear that the radio condition is automatically satisfies for odd i as the difference $f(v_{i+3})-f(v_i)$ exceeds the diameter of Q_n^2. Again for even integer i, $d(v_i,v_{i+3})=\frac{n+1}{4}$ (from Lemma 2.1) and so in this case the radio conditions also hold. On account of all the above all possible cases, we can say that f is an antipodal labelling of Q_n^2. Clearly, the span of f is $\left(\frac{n+9}{4}\right)(2^{n-1}-1)+1$ and it attains at v_{2^n-1}. Therefore, the theorem is proved when $n\equiv 3\ (mod\ 4)$.

Case-2 : $n\equiv 3\ (mod\ 4)$. In this case define a mapping

$$g: V(\ Q_n^2)\rightarrow\{0,1,\dots,\}\text{ by}$$

$$g(v_{i+1})=\begin{cases}\frac{i}{2}\left(\frac{n+3}{4}+1\right), & if\ i\ is\ even\\ \frac{i-1}{2}\left(\frac{n+3}{4}+1\right)+1, & if\ i\ is\ odd\end{cases}$$

To show g is a radio labelling of Q_n^2, we need to show $g(v_j)-g(v_i)\geq\frac{n+1}{2}+1-d(v_i,v_j)$

for all i and j with $j>i$. Observe that if $g(v_j)-g(v_i)\geq\frac{n+1}{2}$ for some i and j , then radio condition automatically satisfies. By simple calculation as used in Case-1, we may show that $g(v_j)-g(v_i)\geq\frac{n+1}{2}$ with $j-i\geq 4$. So we to prove the radio condition for $j\in\{i+1,i+2,i+3\}$. By similar argument with the help of Lemma 4.3, we can show that the radio condition satisfies for these values j.

ISSN(Online): 2320-9801
ISSN (Print): 2320-9798

International Journal of Innovative Research in Computer and Communication Engineering

(An ISO 3297: 2007 Certified Organization)

Vol. 3, Issue 5, May 2015

REFERENCES

1. G. Chartrand, D. Erwin and P. Zhang, 'Radio antipodal colorings of cycles,' Congressus Numer., 144, pp. 129–141, 2000.
2. G. Chartrand, D. Erwin, F. Harrary and P. Zhang, 'Radio labeling of graphs,' Bull. Inst. Combin. Appl., 33, pp. 77–85, 2001.
3. G. Chartrand, D. Erwin and P. Zhang, 'A graph labeling problem suggested by FM channel restrictions,' Bull. Inst. Combin. Appl., 43, pp. 43–57, 2005.
4. S. Das, S.C. Ghosh, S. Nandi and S. Sen, 'A lower bound technique for radio k-coloring,' Discrete Mathematics, 340(5), pp. 855-861, 2017.
5. W.K. Hale, 'Frequency assignment', Theory and application, Proc. IEEE., 68, pp.1497-1514, 1980.
6. J. Jaun and D.D.-F. Liu, 'Antipodal labelings of cycles,' Ars Combin., 103 , pp. 81–86, 2012.
7. R. Khennoufa and O. Togni, 'The radio antipodal and radio numbers of the hypercube', Ars Combin., 102, pp. 447–461, 2011.
8. R. Khennoufa and O. Togni, 'A note on radio antipodal colorigs of paths', Math.Bohem., 130(1), pp 277–282, 2005
9. M. Kchikech, R. Khennoufa, and O. Togni, 'Radio k-labelings for cartesian products of graphs', Electronic Notes in Discrete Mathematics, 22, pp. 347352, 2005.
10. S. R. Kola, and P. Panigrahi, 'Nearly antipodal chromatic number $ac'(P_n)$ of a path P_n', Math.Bohem. 134(1), pp. 77–86, 2009.
11. S. R. Kola, and P, Panigrahi, 'On Radio $(n-4)$ −chromatic number the path P_n,' AKCE Int. J. Graphs Combin., 6(1), pp. 209–217, 2009.
12. S. R. Kola, and P, Panigrahi, 'An improved Lower bound for the radio k-chromatic number of the Hypercube Q_n', Comput. Math. Appl., 60(7), pp. 2131–2140, 2010.
13. D.D.-F. Liu, 'Radio number for trees', Discrete Math., 308 , pp. 1153–1164, 2008.
14. D.D.-F. Liu and M. Xie, 'Radio Number for Square Paths', Ars Combin., 90, pp.307–319, 2009.
15. D.D.-F. Liu and M. Xie, 'Radio number for square of cycles', Congr. Numer., 169, pp. 105–125, 2004.
16. D. Liu and X. Zhu, 'Multi-level distance labelings for paths and cycles', SIAM J. Discrete Math., 19(3), pp. 610–621, 2005.
17. S.R. Kola and P. Panigrahi, 'An improved lower bound for the radio k-chromatic number of the Hypercube Q_n', Compute. Math. Appl., 60(7), pp. 2131–2140, 2010.
18. X. Li, V. Mak and S. Zhou, 'Optimal radio labellings of complete m-ary trees', Discrete Appl. Math., 158, pp. 507–515, 2010.
19. M. Morris-Rivera, M. Tomova, C. Wyels and Y. Yeager, 'The radio number of $C_n \times C_n$,' Ars Combin., 103, pp. 81-96, 2012.
20. J.P. Ortiz, P. Martinez, M. Tomova and C. Wyels, 'Radio numbers of some generalized prism graphs', Discuss. Math. Graph Theory., 31(1), pp. 45–62, 2011.
21. L. Saha and P. Panigrahi, 'On the Radio number of Toroidal grids', Australian J. of Combin., 55, pp. 273–288, 2013.
22. L. Saha and P. Panigrahi, 'A lower bound for radio k-chromatic number', Discrete Applied Mathematics, 192 , pp. 87-100, 2015.
23. L. Saha and P. Panigrahi, 'A Graph Radio k-coloring Algorithm', Lecture notes in Computer Science, 7643 , pp. 125-129, 2012.