A Study of Spline Interpolation

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ABSTRACT: In this paper, we construct A new kind of quintic lacunary g-splines which are the solution of (0, 3, 5)-interpolation problem and error bonds with functions belonging to $C^{(5)}(I)$.Our methods are of lower degree having better convergence property than the earlier investigations using piecewise polynomials with certain specific properties.

KEYWORDS: Spline function, lacunary interpolation, quintic splines piecewise polynomial.

I.INTRODUCTION

Spline interpolation method, as applied to the solution of differential equation employ some from approximating function such as polynomials to approximate the solution by evaluating the function for sufficient number of points in the domain of the solution. Spline functions are a good tool for the numerical approximation of functions on one hand and they suggest new challenging and rewarding problem's on the other hand. Piecewise linear functions as well as step functions have along been important theoretical and practical tools for approximation of function. Lacunary interpolation by splines appears whenever observation gives scattered or irregular information about a function and its derivatives. The data in the problem of lacunary interpolation has also values of the functions and its derivatives but without Hermite conditions that only consecutive derivative is used at each node. Spline function are arise in many problems of mathematical Physics such as viscoelasticity, hydrodynamics, electromagnetic theory, mixed boundary problems in mathematical physics, biology and Engineering.

 Th Fawzy ([3] [4]) constructed special kinds of lacunary quintic g-splines and proved that for functions $f \in C^{(4)}$ the method converges faster that investigated by A.K. Verma[1] and for functions $f \in C^{(5)}$ the order of approximation is the same as the best order of approximation using quintic g- splines. Saxena and Tripathi [7] have studied splines methods for solving the (0,1,3) interpolation problem. They have used spline interpolants of degree six for functions **f** $\in C^{(6)}$ to solved the problem ..R.S.Misra and K.K. Mathur [2] solved lacunary interpolation by splines (0;0 2,3) and(0;0,2,4) cases. During the past twentieth both theories of splines and experiences with their use in numerical analysis have undergone a considerable degree of development. According to Fawzy [3] the interest in spline function is due to the fact that spline function are a good tool for the numerical approximation of functions.The collection of polynomials that form the curve of polynomials that form the curve is collectively referred to as " the spline". The traditional and constrained cubic splines are few different groups of the same family. The group of traditional cubic splines can furthermore be divided into sub group natural, parabolic, runout, cubic run-out and damped cubic splines. The natural cubic spline is by far the most popular and widely used version of the cubic splines family. Spline functions are used in many areas such as interpolation, datafitting, numerical solution of ordinary partial differential equation and also numerical solution of integral equations Lacunary interpolation by splines appears function about a function and its derivatives but without Hermite condition in which consecutive derivatives are used at each nodes, Several researchers have studied the use of spline to solve such interpolation [5 ,8 , 9 ,10 , 11] One uses polynomial for approximation because they can be evaluated. cubic spline interpolation is the most common piecewise polynomial method and is referred as "piecewise" since a unique polynomial is fitted between each pair of data points.

In addition to the paper mentioned above dealing with best interpolation on approximation by splines there were also few papers that deal with constructive properties of space of splines interpolation. In my earlier work [6] [12] [13] [14] some kinds of lacunary interpolation by g-splines have been investigated. In this paper we will continue to discuss the problem.

This paper is organized as follows- In Section 2, we construct a quintic lacunary interpolation (0,3,5) through g-spline. In section 3 we establish the error bonds for interpolatory polynomials for **f** $\in C^{(5)}$ here we also define a

Lemma and theorems about spline functions, by using some specific conditions , the method converges faster than the earlier investigations.

II. SPLINE INTERPOLANT (0, 3, 5) FOR $f \in C^{(5)}$ [I]

Let

$$\Delta: \; 0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$$

be a partition of the interval I = [0,1] with $X_{k+1} - x_k = h_k$, $k = o(1)n - 1$.

and

$s_{1,\Delta}$ be a piecewise polynomial of degree ≤ 5. The spline interpolation(0, 3, 5) for $f \in C^{(5)}$ (I) is given by :

(2.1) $S_{1,\Delta}(x) = s_{1,k}(x) = \Sigma_{J=0}^{5} \dfrac{s_{k,j}^{(1)}}{j!} (x - x_k)^j, x_k \leq x \leq x_{k+1}$, k = O(1)n − 1,

Where $s_{k,j}^{(1)}$'s are explicitly given below in terms of the prescribed data

$\{f_k^{(j)}\}$, j = 0,3,5 ; k = 0(1)n.

In particular for k=0(1)n-1

(2.2) $s_{k,j}^{(1)} = f_k^{(j)}$, j = 0,3,5 for j = 1, 2, 4 , we have

(2.3) $s_{k,2}^{(1)} = \dfrac{4}{h^2}$ $[h \; \{ f_{xk+1}^{(1)} - \dfrac{h^2}{2!} f_{xk}^{(3)} - \dfrac{h^4}{4!} f_{xk}^{(5)}\} - \{f_{xk+1} - f_{xk} - \dfrac{h^3}{3!} f_{xk}^{(3)} - \dfrac{h^5}{5!} f_{xk}^{(5)}\} - \dfrac{h^2}{4}\{f_{xk+1}^{(2)} - hf_{xk}^{(3)} - \dfrac{h^3}{3!} f_{xk}^{(5)}\}]$

(2.4) $s_{k,4}^{(1)} = \dfrac{8}{h^4}$ $[\{ f_{xk+1} - f_{xk} - \dfrac{h^3}{3!} f_{xk}^{(3)} - \dfrac{h^5}{5!} f_{xk}^{(5)}\} - h\{f_{xk+1}^{(1)} - \dfrac{h^2}{2!} f_{xk}^{(3)} - \dfrac{h^4}{4!} f_{xk}^{(5)}\} + \dfrac{h^2}{2} \{f_{xk+1}^{(2)} - hf_{xk}^{(3)} - \dfrac{h^3}{3!} f_{xk}^{(5)}\}]$

(2.5) $s_{k,1}^{(1)} = \dfrac{1}{h}[f_{xk+1} - f_{xk} - \dfrac{h^2}{2!} s_{k,2}^{(1)} - \dfrac{h^3}{3!} f_{xk}^{(3)} - \dfrac{h^4}{4!} s_{k,4}^{(1)} - \dfrac{h^5}{5!} f_{xk}^{(5)}]$

The coefficients $s_{k,j}^{(1)}$ j = 1, 2, 4, have been so choosen

$$D_L^{(p)} S_{1,k} (x_{k+1}) = D_R^{(p)} S_{1,k+1} (x_{k+1}) \; p = 0, 3, 5, \; k = 0(1) \, n$$

Therefore

$S_{1,\Delta} \in C^{(0,3,5)}$ [I] = { f : $f^{(p)} \in C(I), p = 0,3,5$ } is a unique quintic piecewise polynomial satisfying interpolatory conditions (2.2). Using Taylor's expansion in (2.3) to (2.5) for $f \in C^{(5)}$ (I) , we have

(2.6) $\left| s_{k,j}^{(1)} - f_{xk}^{(j)} \right| \leq C_{k,j}^{(1)} h^{5-j} \omega(f^{(5)}; h)$ j = 1,2,4 k=0(1)n-1

Where the constants $C_{k,j}^{(1)}$ are given by

$$C_{k,1}^{(1)} = \dfrac{5}{36} , \; C_{k,2}^{(1)} = \dfrac{1}{5} , \; C_{k,4}^{(1)} = \dfrac{11}{15}$$

Using equation (2.1) to (2.6) and a little computation gives :

Theorem 2.1

Let $f \in C^{(5)}$ (I) and $S_{1,\Delta} \in C^{(0,3,5)}$ [I] be the unique spline interpolant (0, 3, 5) given in (2.1) - (2.5),

then

(2.7) $|| D^{(j)} (f - S_{1,\Delta}) || L_\infty [x_k , x_{k+1}] \leq c_{1,k}^{j} h^{5-j} \omega(f^{(5)}, h)$, j =0(1) 5; k=0 (1) n-1

Where the constants $c_{1,k}^{(j)}$'s are given by :

$c_{1,k}^{0} = \dfrac{5}{18}$, $c_{1,k}^{1} = \dfrac{181}{360}$, $c_{1,k}^{2} = \dfrac{11}{15}$, $c_{1,k}^{(3)} = \dfrac{37}{30}$, $c_{1,k}^{(4)} = \dfrac{26}{15}$, $c_{1,k}^{(5)} = 1$.

III.   ERROR BONDS FOR SPLINE INTERPOLANTS

Suppose $\mathbf{f} \in \boldsymbol{C^{(5)}}$ **[I],** then by the Taylor expansions, we establish the following Lemma by using the modulus of continuity $\omega\left(f^{(5)}; \mathsf{h}\right)$.

Lemma 3.1

For j = 1, 2, and 4   we have

$$\left| S_{k,j}^{(2)} - f_k^{(j)} \right| \leq C_{k,j}^{(2)} h^{5-j} \, \omega\left(f^{(5)}; \mathsf{h}\right),$$

J = 1, 2, and 4.

K = 0(1)n-1

Where the constants $C_{k,j}^{(2)}$ are given by :

$$C_{k,1}^{(2)} = \frac{5}{36}, \quad C_{k,2}^{(2)} = \frac{1}{5}, \quad C_{k,4}^{(2)} = \frac{11}{15},$$

Proof.

For j = 1, 2, and 4   Using Taylor's expansion from (2.1)-(2.5), we have

(3.1) $\left| S_{k,1}^{(2)} - f_k^{(1)} \right| \leq \frac{5}{36} h^4 \, \omega\left(f^{(5)}; \mathsf{h}\right),$

(3.2) $\left| S_{k,2}^{(2)} - f_k^{(2)} \right| \leq \frac{1}{5} h^3 \, \omega\left(f^{(5)}; \mathsf{h}\right),$

and

(3.3) $\left| S_{k,4}^{(2)} - f_k^{(4)} \right| \leq \frac{11}{15} \mathsf{h} \, \omega\left(f^{(5)}; \mathsf{h}\right),$

This completes the Proof of the Lemma 3.1

Theorem 3.1

Let $f \in C^{(5)}(I)$ and $S_{2,\Delta} \in C^{(0,4,5)}$ [I]   be the unique spline interpolant (0, 3, 5) given in (2.1) - (2.5),

then

(3.5) $\left\| D^{(j)}(f\text{-} S_{2,\Delta}) \right\| L_\infty [x_k, x_{k+1}] \leq c_{2,k}^j h^{5-j} \omega(f^{(5)}, h),$ j =0(1) 5;   k=0 (1) n-1

Where the constants $c_{2,k}^{(j)}$'s are given by :

$$c_{2,k}^{(0)} = \frac{5}{18}, \quad c_{2,k}^{(1)} = \frac{181}{360}, \quad c_{2,k}^{(2)} = \frac{11}{15}, \quad c_{2,k}^{(3)} = \frac{37}{30}, \quad c_{2,k}^{(4)} = \frac{26}{15} \quad c_{2,k}^{(5)} = 1.$$

Proof :

For k = 0(1)n-1 , j = 0(1)5

$\left| f(x) - S_{2,\Delta} \right| \leq \left| f(x) - S_k(x) \right|$

$\leq \Sigma_{J=0}^4 \frac{|f^{(j)}(\mathsf{x}k) - s_k^{(j)}| \, \mathsf{h}^{(j)}}{j!} + \frac{|f^{(5)}(\delta k) - s_k^{(5)}| \, \mathsf{h}^{(5)}}{5!}$

Where $x_k < S_k < x_{k+1}$ Using Lemma 3.1 and the definition of the modulus of continuity of $\boldsymbol{f^{(5)}}$(x), we obtain

(3.6) $\left| f(x) - S_k(x) \right| \leq \frac{5}{18} h^5 \, \omega\left(f^{(5)}; \mathsf{h}\right),$

(3.7) $\left| f_{(x)}^{(1)} - S_k^{(1)}(x) \right| \leq \frac{181}{360} h^4 \, \omega\left(f^{(5)}; \mathsf{h}\right),$

(3.8) $\left| f_{(x)}^{(2)} - S_k^{(2)}(x) \right| \leq \frac{11}{15} h^3 \, \omega\left(f^{(5)}; \mathsf{h}\right),$

(3.9) $\left| f_{(x)}^{(3)} - S_k^{(3)}(x) \right| \leq \frac{37}{30} h^2 \, \omega\left(f^{(5)}; \mathsf{h}\right),$

(3.10) $\left| f_{(x)}^{(4)} - S_k^{(4)}(x) \right| \leq \frac{26}{15} \mathsf{h} \, \omega\left(f^{(5)}; \mathsf{h}\right),$

and

(3.11) $\left| f_{(x)}^{(5)} - S_k^{(5)}(x) \right| \leq \omega\left(f^{(5)}; \mathsf{h}\right),$

Using (3.6)-(3.11) , completes the Proof of the Theorem 3.1.

IV.CONCLUSION

In this paper, we have studied the existence and uniqueness of quintic lacunary interpolation (0 ,3 , 5) and error bonds for Interpolants for functions belonging to $C^{(5)}(I)$. Our methods are of lower degree having better convergence property. Also we conclude that this new technique we used in proving of the Lemma and one important theorem of spline function is far more better than the earlier investigations.

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