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Operations on Digraphs and Digraph Folding

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ABSTRACT: In this paper we examined the relation between digraph folding of a given pair of digraphs and digraph folding of new digraphs generated from these given pair of digraphs by some known operations like union, intersection, joins, Cartesian product and composition. We first redefined these known operations for digraphs, then we defined some new maps of these digraphs and we called these maps union, intersection, join, Cartesian and composition dimaps. In each case we obtained the necessary and sufficient conditions, if exist,for a dimap to be digraph folding. Finally we explored the digraph folding, if there exist any, by using the adjacency matrices.

KEY WORDS: Digraphs, adjacency matrices,digraph folding,union,intersection,join the Cartesian product and the composition of digraphs

I. INTRODUCTION

Graph folding is introduced by E.EL-Kholy and A.AL-Esway [3]. The notion of digraph folding is introduced by E.EL-Kholy and H.Ahmed [4]. Definitions (1.1)

(1) A digraph D consists of elements, called vertices and a list of ordered pairs of these elements, called arcs. The set of vertices is called the vertex set of D, denoted by V (D), and the list of arcs is called the arc list of D, denoted by A (D). If v and we are vertices of D, then an arc of the form vw is said to be directed from v to w. The digraph with no loops is called simple. Two or more arcs joining the same pair of vertices in the same direction is called multiple arcs [5].

(2) Let D_1 and D_2 be digraphs and f: $D_1 \to D_2$ a continuous function. Then f is called a digraph map if,

(i)For each vertex v ϵV (D_1), f(v) is a vertex in V (D_2).

(ii)For each arc e ϵ A (D_1), dim (f(e)) $\leq$ dim (e) [4].

(3) Let D_1 and D_2 be simple digraphs, we call a digraph map f: $D_1 \to D_2$ a digraph folding if f maps vertices to vertices and arcs to arcs, i.e., for each v ϵ V (D_1), f(v) ϵ V (D_2) and for each e ϵ A (D_1), f(e) ϵ A (D_2)(4) If the digraph contains loops, then the digraph folding must send loops to loops but of the same direction. Theset of digraph folding between digraphs D_1 and D_2 is denoted by Đ(D_1,D_2) and from D into itself by Đ(D).

(5)Let D be a diagraph without loops, with n vertices labeled 1, 2, 3,…, n. The adjacency matrix M(D) is the nxn matrix in which the entry in row i and column j is thenumber of arcs from vertex i to vertex j [5].

A. Proposition

Let D be a connected digraph without loops with n vertices. Then a digraph folding of D into itself may be defined, if there is any, as a digraph map f of D to an image f(D) by mapping:

(i) The multiple arcs into one of its arcs.

(ii)(a)The vertex v_i to the vertex v_j if the numbers appearing in the adjacencymatrix in the i^{th} and j^{th} rows (or columns) are the same.

(b)The vertex v_i to the vertex v_j if the entries of the i^{th} and j^{th} rows are zerosandif the i^{th} and j^{th} columns are the same, or there exists a row k which has numbers 1 in the i^{th} and j^{th} columns.

(iii)(a)The arc (v_i,v_k) to the arc (v_j,v_k) if the i^{th} and j^{th} rows (or columns) arethesame.

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(b) The arc (v_i,v_j) to the arc (v_i, v_k) if the j^{th} and k^{th} columns (or rows) are the same.
In general the arc (v_i,v_j) will be mapped to the arc (v_k,v_l) if v_i mapped to v_k and v_j mapped to v_l, [4].

II. UNION OF DIGRAPHS

In the following we redefine the known operation, union, given for two simple graphs [3], for digraphs.
Definition (2-1)
Let $D_1 =(V_1,A_1)$ and $D_2 =(V_2,A_2)$ be simple digraphs .Then the simple digraphD = (V, A) where $V=V_1UV_2$ and $A=A_1UA_2$ is called the union of digraphs D_1 and D_2 and is denoted by D_1UD_2 .When D_1 and D_2 are vertex disjoint $D_1UD_2$is denoted by , D_1+D_2 , and is called the sum of digraphs $D_1$and D_2.
Definition (2-2)
Let $D_1 =(V_1,A_1)$ and $D_2 =(V_2,A_2)$ be simple digraphs .Let f: D_1D_1 andg:D_2D_2 be digraph maps.By the union dimap of the digraph maps f and g, fUg,we mean a digraph map from the digraph D_1UD_2 into itself.f Ug:$D_1UD_2D_1UD_2$such thatf(v) = g(v) , for all v ϵ $V_1 \cap V_2$, f(e)=g(e) , for all e ϵ $A_1 \cap A_2$defined by

$$\text{(i) For each } v \in V_1 \cup V_2,\ (fUg)(v) = \begin{cases} f(v), & \text{if } v \in V_1 \\ g(v), & \text{if } v \in V_2 \end{cases}$$

$$\text{(ii) For each } e \in A_1UA_2,\ (fUg)(e) = \begin{cases} f(e), & \text{if } e \in A_1 \\ g(e), & \text{if } e \in A_2 \end{cases}$$

A. Theorem
Let D1=(V1,A1) and D2=(V2, A2) be simple connected digraphs .Let f:D1 D1 and g:D2 D2 be digraph maps. Then the union dimap fUg is a digraph folding if f and g are digraph foldings. In this case (fUg)(D1UD2) =f(D1) U g(D2) The proof is almost as in [2]. Example 2.4
Let D1=(V1,A1),where V1={v1,v2,v3,v4} and A1={e1,e2,e3,e4,e5}.
Let D2= (V2,A2), where V2={v1, v2, v3, v5} and A2= {e1, e2, e6}, see Figure1.

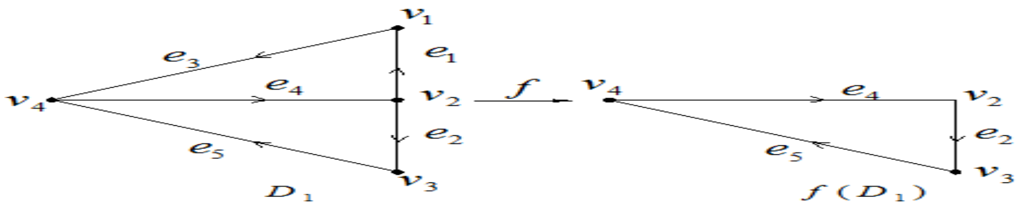
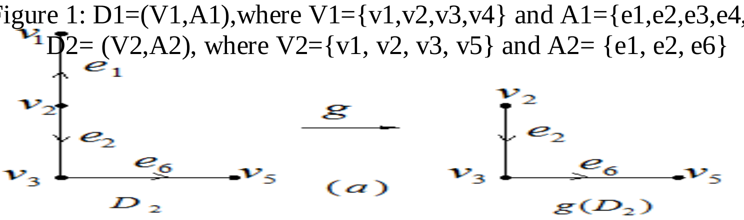


Figure 1: D1=(V1,A1),where V1={v1,v2,v3,v4} and A1={e1,e2,e3,e4,e5}
D2= (V2,A2), where V2={v1, v2, v3, v5} and A2= {e1, e2, e6}



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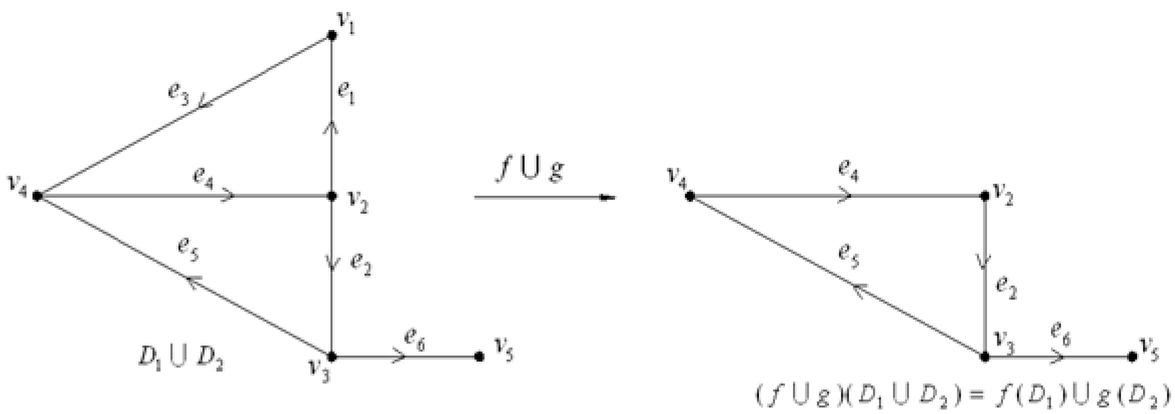

Figure 2: fUg: D1UĐ2 Đ1UD2

Now let f ϵĐ(D_1) be a digraph folding defined by f{v1}={v3} and f{e1,e3}={e2,e5}, where through this paper the omitted vertices and arcs will be mapped to themselves . Also, let g ϵ Đ(D_2)be a digraph folding defined by g{v1}={v3} and g{e1}={e2} , see Figure1.The union dimapfUg: D1UD2D1UD2 defined by (fUg){v1}={v3} and (fUg){e1,e3}= {e2,e5} is a digraph folding , see Figure2.The adjacency matrices of D1,D2 and D1UD2 are as follows :

$$M(D_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, M(D_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_5 \end{array} \begin{bmatrix} v_1 & v_2 & v_3 & v_5 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } M(D_1 \cup D_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By using only these adjacency matrices we can define the digraph folding .Forexample ,by using the adjacency matrix $M(D_1)$ we can easily see that the vertex v_1 will be mapped to the vertex v_3 since the first and third rows of $M(D_1)$ have the same entries .Also the arc $(v_1,v_4)=e_3$ will be mapped to the arc$(v_3,v_4)= e_5$ since the 1^{st} and 3^{rd} rows are the same, finally the arc $(v_2,v_1)=e_1$ will be mapped to the arc $(v_2,v_3)= e_3$ since the 1^{st} and 3^{rd} columns are the same. Again by using $M(D_2)$ and $M(D_1UD_2)$ we can describe the digraph folding of both D_2 and D_1UD_2

III. INTERSECTION OF DIGRAPHS

A. Definition

Let $D_1 = (V_1,A_1)$ and $D_2 = (V_2,A_2)$ be simple digraphs . Then the simple digraph D =(V,A) where $V = V_1 \cap V_2$ and $A = A_1 \cap A_2$ is called the intersection of digraphs D_1 and D_2 and is denoted by $D_1 \cap D_2$.

B. Definition

Let $D_1=(V_1,A_1)$ and $D_2 =(V_2,A_2)$ be simple digraphs .Let f : D_1D_1 andg:D_2D_2 be digraph maps. If f and g agree on $V_1 \cap V_2$ and $A_1 \cap A_2$ then by the intersection dimap of the digraph maps f and g , f ∩g,we mean a digraph mapf∩g: $D_1 \cap D_2 D_1 \cap D_2$, where$V_1 \cap V_2 \neq \emptyset$defined by:

(i) For all v ϵ $V_1 \cap V_2$, (f ∩g)(v) = f(v) = g(v)

(ii) For all e ϵ $A_1 \cap A_2$,(f ∩g) (e) = f(e)= g(e)

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Let D1=(V1,A1) and D2=(V2,A2) be simple connected digraphs .Let f:D1 D1 and g:D2 D2 be digraph maps .Then the intersection dimapf∩g is a digraph folding if f and g are digraphfolding. Inthiscase(f∩g)(D1∩D2)=f(D1)∩g(D2). The proof is easy.

let $D_1=(V_1,A_1)$,where $V_1=\{ v_1,v_2,v_3,v_4,v_5,v_6 \}$ and $A_1=\{e_1,e_2,e_3,e_4,e_5,e_6,e_7,e_8\}$.Let $D_2=(V_2,A_2)$, where $V_2=\{ v_1,v_5,v_6,v_7\}$ and $A_2=\{ e_5,e_8,e_9,e_{10},e_{11} \}$,see Figure 3.

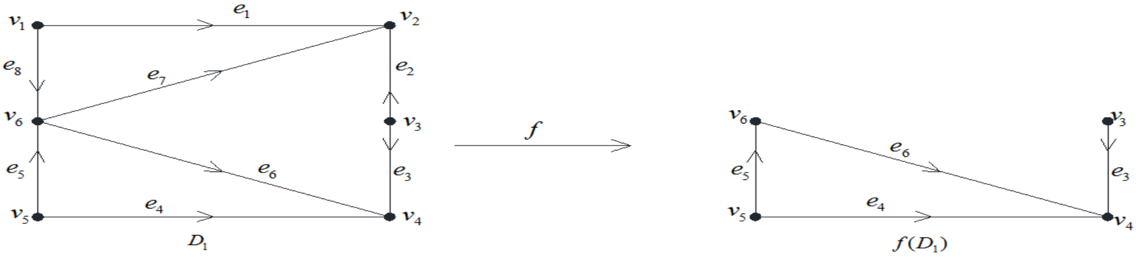

Figure 3: $D_1=(V_1,A_1)$,where $V_1=\{ v_1,v_2,v_3,v_4,v_5,v_6 \}$ and $A_1=\{e_1,e_2,e_3,e_4,e_5,e_6,e_7,e_8\}$

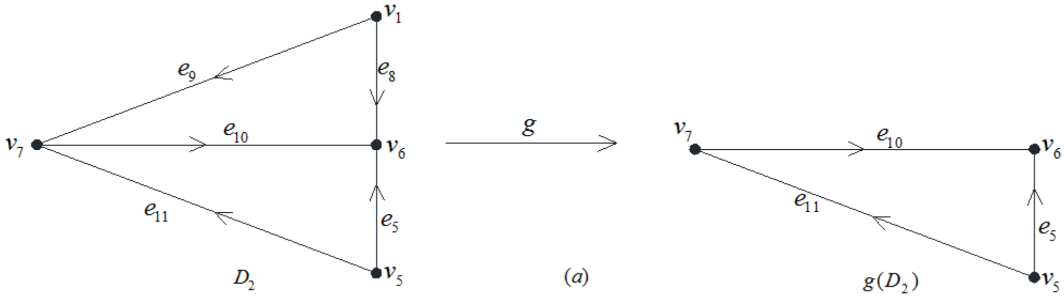

Figure 4:$D_2=(V_2,A_2)$, where $V_2=\{ v_1,v_5,v_6,v_7\}$ and $A_2=\{ e_5,e_8,e_9,e_{10},e_{11} \}$

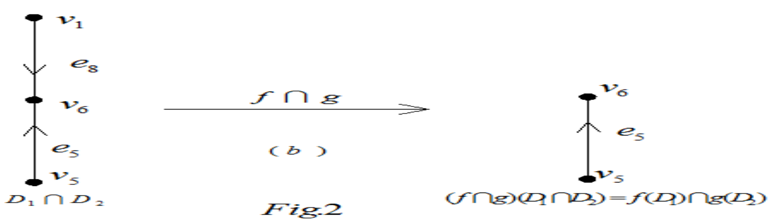

Figure 5: (f ∩ g){v1}={v5} and (f ∩g){e8}={e5}

Now let $f \epsilon Đ(D_1)$ be a digraph folding defined by $f\{v_1,v_2\} = \{v_5,v_4\}$ and$f\{e_1,e_2,e_7,e_8\}=\{e_4,e_3,e_6,e_5\}$.Also , let $g \epsilon Đ(D_2)$be a digraph folding defined by $g\{v_1\}=\{v_5\}$ and $g\{e_8,e_9\}=\{e_5,e_{11}\}$, see Figure 1.The intersection dimapf ∩g :$D_1 \cap D_2 D_1 \cap D_2$ defined by $(f \cap g)\{v_1\}=\{v_5\}$ and $(f \cap g)\{e_8\}=\{e_5\}$ is a digraph folding, see Figure 2.The adjacency matrices of D_1,D_2 and $D_1 \cap D_2$ are as follows:

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$$M(D_1)=\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6\end{array}\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6\\ 0&1&0&0&0&1\\ 0&0&0&0&0&0\\ 0&1&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&1\\ 0&1&0&1&0&0\end{bmatrix},\ M(D_2)=\begin{array}{c} \\ v_1\\ v_5\\ v_6\\ v_7\end{array}\begin{bmatrix} v_1 & v_5 & v_6 & v_7\\ 0&0&1&1\\ 0&0&1&1\\ 0&0&0&0\\ 0&0&1&0\end{bmatrix}\ and\ M(D_1\cap D_2)=\begin{array}{c} \\ v_1\\ v_5\\ v_6\end{array}\begin{bmatrix} v_1 & v_5 & v_6\\ 0&0&1\\ 0&0&1\\ 0&0&0\end{bmatrix}$$

By using only these adjacency matrices we can define the digraph folding. For example, by using the adjacency matrix $M(D_1 \cap D_2)$ we can easily see that the vertex v_1 will be mapped to the vertex v_5, since the first and second rows have the same entries. Also, the arc $(v_1,v_6)=e_8$ can be mapped to the arc $(v_5,v_6)=e_5$, since the first and second rows are the same. Again by using $M(D_1)$ and $M(D_2)$ we can describe the digraphfoldingof both D_1 and D_2.

IV. JOIN OF DIAGRAPHS

A. Definition

Let D_1 and D_2 be vertex dis joint diagraphs. Then we define the join digraph, D_1 v D_2, to be the digraph in which each vertex of D_1 or D_2 is adjacent to the vertices of D_2 (or D_1).

B. Definition

Let D1=(V1,A1), D2=(V2,A2),D3=(V3,A3) and D4=(V4,A4) be simple digraphs. Let f: D1 ⟶ D3 and g: D2 ⟶ D4 be digraph maps. By a join dimap, we mean a digraph map, fvg: D1vD2 ⟶ D3vD4 defined by

(i) For each vertex $v \in V_1 \cup V_2$, $(fvg)(v)=\begin{cases} f(v), & \text{if } v \in V_1\\ g(v), & \text{if } v \in V_2\end{cases}$

(ii) For each arc $e=(v_1,v_2), v_1 \in V_1$ and $v_2 \in V_2$, $(fvg)\{e\}=\{f(v_1),g(v_2)\} \in A_3 v A_4$.

(iii) If e=(u1,v1) ϵ A1, then (fvg){e}=(fvg){(u1,v1)}={f(u1),f(v1)}, Also if e =(u2,v2) ϵ A2 ,then (fvg){e}=(fvg){(u2,v2)}={ g(u2),g(v2)}

Note that If $f\{u_1\}= f\{v_1\}$,then the image of the join dimap (fvg){e} will be a vertex of D_3 v D_4 and thus is not a digraph folding.

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Let D1, D2, D3and D4 be digraphs, let f:D1 ⟶ D3 and g:D2 ⟶ D4 be digraph maps .Then (fvg) ϵ Đ(D1vD2, D3vD4) is a digraph folding if f and g are digraph folding.

Proof: Suppose f and g are digraph folding.Then (fvg){ V1UV2 }= {f(V1)Ug(V2)}.But f(V1) ϵ V(D3) , g(V2) ϵ V(D4).Thus{f(V1)Ug(V2)} ϵ V(D3vD4) , i.e., fvg maps vertices to vertices .Now, let e ϵ A(D1vD2).Then either e ϵ A(D1) or e ϵ A(D2) or e is an arc joining a vertex of D1 (or D2) to a vertex of D2 (or D1).In the first two cases and since each of f and g is a digraph folding,(fvg){e}ϵA(D3vD4).Now, if e=(v1,v2), v1 ϵ D1 and v2 ϵ D2.Then(fvg){e}=(fvg){(v1,v2)}={f(v1),g(v2)}={(v3,v4)} ϵ A(D3vD4). Thus fvg maps arcs to arcs and hence the join digraph map is a digraph folding. The converse is guaranteed by the definition of the join digraph.

C. Example

Let $D_1=(V_1,A_1)$, where $V_1=\{v_1,v_2,v_3,v_4\}$ and $A_1=\{e_1,e_2,e_3,e_4\}$ and $D_2=(V_2,A_2)$, where $V_2=\{v_5,v_6,v_7\}$ and $A_2=\{e_5,e_6\}$, see Figure 3.

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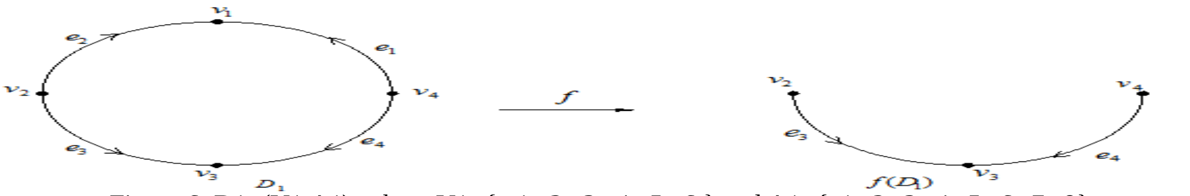

Figure 6: D1=(V1,A1), where V1={ v1,v2,v3,v4,v5,v6 } and A1={e1,e2,e3,e4,e5,e6,e7,e8}

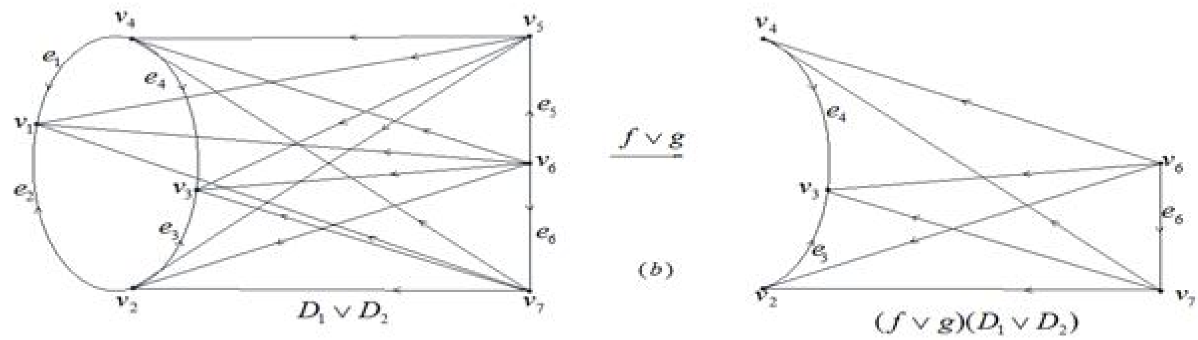

Figure 7: D1= (V1, A1), where V1= {v1, v2, v3, v4} and A1= {e1, e2, e3, e4, e5}
D2= (V2, A2), where V2= {v1, v2, v3, v5} and A2= {e1, e2, e6}

Let f ϵĐ(D_1) be defined by f{v1}={v3} and f{e1,e2}={e4,e3}.Also, let g ϵĐ(D_2) be defined by g{v5}={v7} and g{e5}={e6}.The join dimap fvg:D1v D2 ⟶ D1v D2 is defined by (fvg){v1,v5}={v3,v7}and (fvg){e1}=(fvg){(v4,v1)}={(v4,v3)}={e4},also,(fvg){e5}=(fvg){(v6,v5)}={(v6,v7)}={e6}and (fvg) {(v5,v1)}={(v7,v3)},and so on, see Figure 4.The adjacency matrices of D1, D2 and D1vD2 are as follows:

$$M(D_1)=\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\end{array}\begin{bmatrix} v_1 & v_2 & v_3 & v_4\\ 0&0&0&0\\ 1&0&1&0\\ 0&0&0&0\\ 1&0&1&0\end{bmatrix}, M(D_2)=\begin{array}{c} \\ v_5\\ v_6\\ v_7\end{array}\begin{bmatrix} v_5 & v_6 & v_7\\ 0&0&0\\ 1&0&1\\ 0&0&0\end{bmatrix} \text{ and } M(D_1\cup D_2)=\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6\\ v_7\end{array}\begin{bmatrix} v_1&v_2&v_3&v_4&v_5&v_6&v_7\\ 0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0\\ 1&1&1&1&0&0&0\\ 1&1&1&1&1&0&1\\ 1&1&1&1&0&0&0\end{bmatrix}$$

By using these adjacency matrices we can describe the digraph folding. The adjacency matrix $M(D_1)$ suggests that the vertex v_1 can be mapped to the vertex v_3 since the first and third columns of $M(D_1)$ have the same entries.Also the arc $(v_4,v_1)= e_1$ can be mapped to the arc$(v_4,v_3)=e_4$and the arc$(v_2,v_1)=e_2$can be mapped to the arc$(v_2,v_3)=e_3$ since the 1st and 3rd columns are the same.Again by using $M(D_2)$and $M(D_1vD_2)$we can describe the digraph folding of both D_2 and D_1vD_2.

V. THE CARTESIAN PRODUCT OF DIAGRAPH

A. Definition

The Cartesian product D_1xD_2 of two simple diagraphs is a simple diagraph with vertex set $V(D_1xD_2)=V_1xV_2$ and arc set $A(D_1xD_2)=[(A_1xV_2)U(V_1xA_2)]$such that two vertices (u_1,u_2) and (v_1,v_2) are adjacent in D_1xD_2if,either

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(i) $u_1 = v_1$ and u_2 is adjacent to v_2 in D_2 , or
(ii) u_1 is adjacent to v_1 in D_1 , $u_2 = v_2$.

B. Definition

Let D1, D2, D3 and D4 be simple digraphs .Let f: D1 ⟶ D3 and g: D2⟶ D4 is diagraph maps. Then by the Cartesian product dimapfxg: D1xD2 D3 x D4. ⟶
We mean a dimap defined as follows:
(i)If $v=(v_1,v_2) \in V_1xV_2$, $v_1\in V_1$, $v_2\in V_2$,then $(fxg)(v)=(fxg)(v_1,v_2)=(f(v_1),g(v_2)) \in V_3xV_4$
(ii)If the arc $e=\{(\{v_1\}_i, \{v_2\}_j),(\{v_1\}_i ,\{v_2\}_k)\}$, where $\{v_1\}_i \in V(D_1)$ and $\{v_2\}_j ,\{v_2\}_k \in V(D_2)$, then $(fxg)\{e\}=(fxg)\{(\{v_1\}_i ,\{v_2\}_j) ,(\{v_1\}_i,\{v_2\}_k)\} =\{(\{v_1\}_i, g\{v_2\}_j),(\{v_1\}_i, g\{v_2\}_k)\}$.
(iii)If the arc $e=\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_k ,\{v_2\}_j)\}$, where $\{v_1\}_i ,\{v_1\}_k\in V(D_1)$ and $\{v_2\}_j\in V(D_2)$, then$(fxg)\{e\}=(fxg)\{(\{v_1\}_I,\{v_2\}_j),(\{v_1\}_k,\{v_2\}_j)\}=\{(f\{v_1\}_i,\{v_2\}_j),(f\{v_1\}_k ,\{v_2\}_j)\}$.
Note that if $g\{v_2\}_j=g\{v_2\}_k$or $f\{v_1\}_i=f\{v_1\}_k$,the image of the arc will be a vertex

THEOREM

Let D_1, D_2, D_3 and D_4 be digraphs, letf:D_1D_3 ⟶and g :D_2D_4be digraph maps.Then$(fxg) \in Đ(D_1xD_2,D_3xD_4)$ is a digraph folding iff $\in Đ(D_1,D_3)$ and g $\in Đ(D_2,D_4)$ are digraph foldings.In this case $(fxg)(D_1xD_2)=f(D_1) \times g(D_2)$.
Proof:Suppose f and g are digraph folding. Then for each vertex $(v_1,v_2) \in(D_1xD_2)=V_1xV_2$,$(fxg)\{(v_1,v_2)\}=\{(f(v_1),g(v_2))\}=(v_3,v_4) \in V(D_1xD_2)=V_3 xV_4$, i.e., vertices to vertices. Now , let $e \in A(D_1 xD_2)$, then if $e=\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_k,\{v_2\}_j)\}$,where $\{v_1\}_i$ is adjacent to $\{v_1\}_k$ in $D_1$and $\{v_2\}_j\in D_2$, then $(fxg)\{e\}=(fxg)\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_k,\{v_2\}_j)\}=\{(f\{v_1\}_i,\{v_2\}_j),(f\{v_1\}_k,\{v_2\}_j)\}$, since $\{v_1\}_i$ is adjacent to $\{v_1\}_k$ and f is a digraph folding,Then $f\{v_1\}_i\neq f\{v_1\}_k$.Thus $(fxg)\{e\} \in A(D_3xD_4)$.By the same procedure,if $e=\{(\{v_1\}_i ,\{v_2\}_j),(\{v_1\}_i,\{v_2\}_k)\}$, where $\{v_1\}_i\in V(D_1)$ and $\{v_2\}_j$ is adjacent to$\{v_2\}_k$in D_2,then $(fxg)\{e\} \in A(D_3xD_4)$i.e., fxg maps arcs to arcs and hence the Cartesian product dimap is a digraph folding.To prove the converse suppose that (fxg) is a digraph folding and for g,is not a digraph folding.In this case f or g,will maps an arc to a vertex,say $f\{(u_1,v_1)\}=\{u_3\} \in V(D_3)$.Then $(fxg)\{(\{u_1\},\{v_2\}_j),(\{v_1\},\{v_2\}_j)\}=\{(f\{u_1\},\{v_2\}_j),(f\{v_1\},\{v_2\}_j)\}=\{(\{u_3\},\{v_2\}_j),(\{u_3\},\{v_2\}_j)\} \in V(D_3xD_4)$.This contradicts the assumption and thus each of f and g must be a digraph folding.

A. Examples

(a) Let $D_1=(V_1,A_1)$, where $V_1=\{u_1,u_2,u_3,u_4\}$, $A=\{e_1,e_2,e_3,e_4\}$and $D_2=(V_2,A_2)$, where $V_2 = \{v_1,v_2,v_3\}$, $A= \{e_5,e_6\}$, see Figure 4.

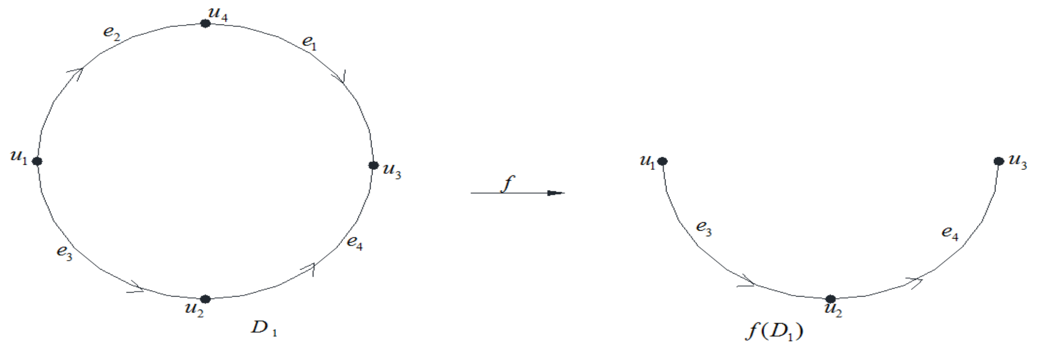

Figure 8: $(fxg)\{(u_4,v_1),(u_4,v_2),(u_4,v_3),(u_1,v_1),(u_2,v_1),(u_3,v_1)\}=\{(u_2,v_1),(u_2,v_2),(u_2,v_3), (u_1,v_3),(u_2,v_3),(u_3,v_3)\}$

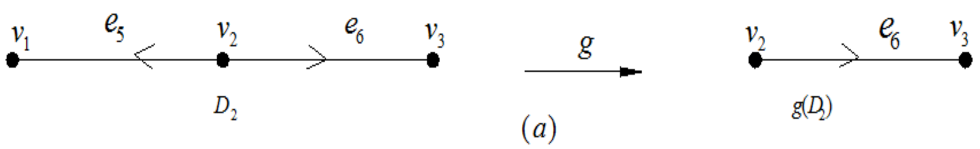

Figure 9: (fvg){e5}= (fvg){(v6,v5)}={(v6,v7)}= {e6}

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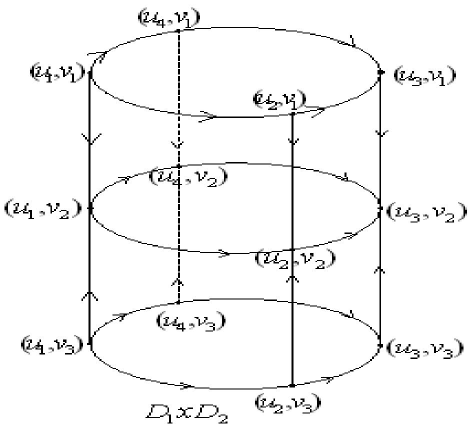

Figure 10: (fxg){((u4,v1),(u3,v1)),((u3,v1),(u3,v2))}= {((u2,v3),(u3,v3)),((u3,v3),(u3,v2))}

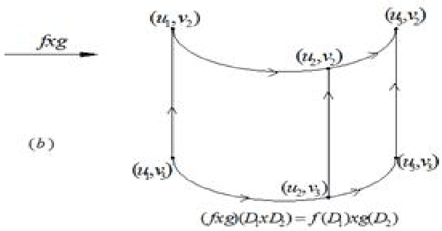

Figure 11: (fxg){((u4,v1),(u3,v1)),((u3,v1),(u3,v2))}= {((u2,v3),(u3,v3)),((u3,v3),(u3,v2))}

Let f $\epsilon Ð(D_1)$ defined by f{u_4}={u_2} and f{e_1,e_2}={e_4,e_3}.Also,let g $\epsilon Ð(D_2)$defined by g{v_1} = {v_3}and g{e_5} = {e_6}.Then the Cartesian product dimapfxg :$D_1xD_2D_3xD_4$is defined as follows :
(fxg){$(u_4,v_1),(u_4,v_2),(u_4,v_3),(u_1,v_1),(u_2,v_1),(u_3,v_1)$}={$(u_2,v_1),(u_2,v_2),(u_2,v_3)$, $(u_1,v_3),(u_2,v_3),(u_3,v_3)$} .Also, (fxg){$((u_4,v_1),(u_3,v_1)),((u_3,v_1),(u_3,v_2))$}={$((u_2,v_3),(u_3,v_3)),((u_3,v_3),(u_3,v_2))$} and so on, see Figure 4. The adjacency matrices of D_1, D_2 and D_1xD_2 are as follows:

$$M\left(D_1\right)=\begin{matrix} & u_1 & u_2 & u_3 & u_4 \\ u_1 & 0 & 1 & 0 & 1 \\ u_2 & 0 & 0 & 1 & 0 \\ u_3 & 0 & 0 & 0 & 0 \\ u_4 & 0 & 0 & 1 & 0 \end{matrix},\ M\left(D_2\right)=\begin{matrix} & v_1 & v_2 & v_3 \\ v_1 & 0 & 1 & 0 \\ v_2 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 0 \end{matrix}\ and$$

$M(D_1xD_2)$

	(u_1,v_1)	(u_2,v_1)	(u_3,v_1)	(u_4,v_1)	(u_1,v_2)	(u_2,v_2)	(u_3,v_2)	(u_4,v_2)	(u_1,v_3)	(u_2,v_3)	(u_3,v_3)	(u_4,v_3)
(u_1,v_1)	0	1	0	1	1	0	0	0	0	0	0	0
(u_2,v_1)	0	0	1	0	0	1	0	0	0	0	0	0
(u_3,v_1)	0	0	0	0	0	0	1	0	0	0	0	0
(u_4,v_1)	0	0	1	0	0	0	0	1	0	0	0	0
(u_1,v_2)	0	0	0	0	0	1	0	0	0	0	0	0
(u_2,v_2)	0	0	0	0	0	0	1	0	0	0	0	0
(u_3,v_2)	0	0	0	0	0	0	0	0	0	0	0	0
(u_4,v_2)	0	0	0	0	0	0	1	0	0	0	0	0
(u_1,v_3)	0	0	0	0	1	0	0	0	0	1	0	1
(u_2,v_3)	0	0	0	0	0	1	0	0	0	0	1	0
(u_3,v_3)	0	0	0	0	0	0	1	0	0	0	0	0
(u_4,v_3)	0	0	0	0	0	0	0	1	0	0	1	0

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Once again we can describe the digraph foldings by using $M(D_1)$,$M(D_2)$ and $M(D_1XD_2)$.For example, from $M(D_1XD_2)$ we can see that the vertex (u_4,v_1) can be mapped to the vertex (u_2,v_1) since the second and fourth columns are the same . Also, the arc $((u_1,v_1),(u_4,v_1))$ will be mapped to the arc $((u_1,v_3),(u_2,v_3))$ since the vertex (u_4,v_1) is mapped to the vertex (u_2,v_3) and the vertex (u_1,v_1) is mapped to the vertex (u_1,v_3), and so on, see Figure 4.

(b)Let $D_1=(V_1,A_1)$, where $V_1=\{u_1,u_2,u_3,u_4\}$, $A_1=\{e_1,e_2,e_3,e_4\}$ and $D_2=(V_2,A_2)$, where $V_2 = \{v_1,v_2,v_3,v_4\}$, $A_2 = \{e_5,e_6,e_7,e_8\}$, see Figure 5.

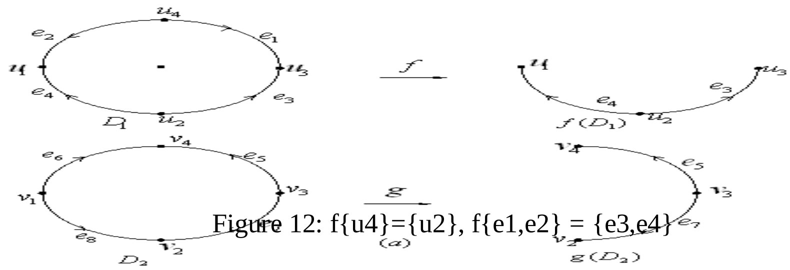

Figure 12: f{u4}={u2}, f{e1,e2} = {e3,e4}

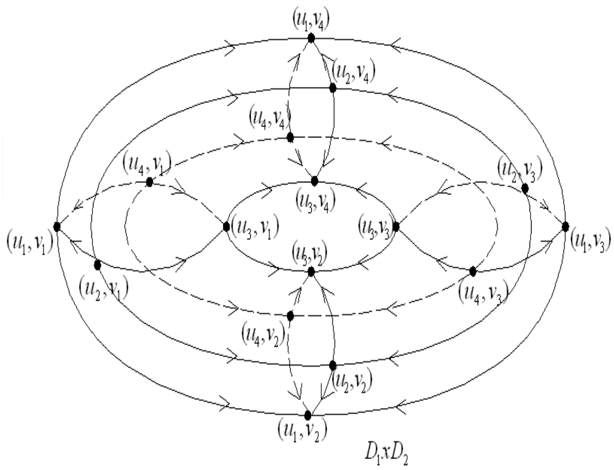

Figure 13: (fxg){((u4,v1),(u3,v1)),((u3,v1),(u3,v2))}= {((u2,v3),(u3,v3)),((u3,v3),(u3,v2))}

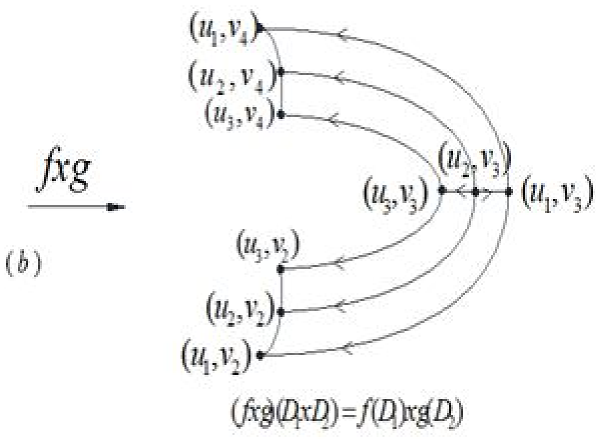

Figure 14: (fxg){((u4,v1),(u3,v1)),((u3,v1),(u3,v2))}={((u2,v3),(u3,v3)),((u3,v3),(u3,v2))}

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Let f $\epsilon Đ(D_1)$be defined by f{u_4}={u_2}, f{e_1,e_2} = {e_3,e_4} and g $\epsilon Đ(D_2)$ be defined by g{v_1} = { v_3} , g{e_6 ,e_8}= {e_5,e_7}. Then the cartesian product dimap h = f x g : D_1 x D_2 → D_1 x D_2 is defined as follows: h{(u_4,v_2),(u_3,v_1)}={(u_2,v_2),(u_3,v_3)}, and so on. Also, h{((u4,v2),(u1,v2)),((u3,v1),(u3,v2))}= {((u2,v2),(u1,v2)),((u3,v3),(u3,v2))}, and so on, see Figure 6. The adjacency matrices of D1,D2 and D1 x D2 are as follows:

$$M(D_1) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \quad , \quad M(D_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \text{ and}$$

$M(D_1 x D_2)$ =

	(u_1,v_1)	(u_2,v_1)	(u_3,v_1)	(u_4,v_1)	(u_1,v_2)	(u_2,v_2)	(u_3,v_2)	(u_4,v_2)	(u_1,v_3)	(u_2,v_3)	(u_3,v_3)	(u_4,v_3)	(u_1,v_4)	(u_2,v_4)	(u_3,v_4)	(u_4,v_4)
(u_1,v_1)	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
(u_2,v_1)	1	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0
(u_3,v_1)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
(u_4,v_1)	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	1
(u_1,v_2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(u_2,v_2)	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
(u_3,v_2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(u_4,v_2)	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
(u_1,v_3)	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
(u_2,v_3)	0	0	0	0	0	1	0	0	1	0	1	0	0	1	0	0
(u_3,v_3)	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
(u_4,v_3)	0	0	0	0	0	0	0	1	1	0	1	0	0	0	0	1
(u_1,v_4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(u_2,v_4)	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0
(u_3,v_4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(u_4,v_4)	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0

Once again we can describe the digraph foldings by using M(D_1) and M(D_1 xD_2).For example, from M(D_1 x D_2) we can see that the vertex (u_4,v_2) can be mapped to the vertex (u_2,v_2) since the 6^{th} and 8^{th} rows have the same entries. And the vertex (u_3,v_1) can be mapped to the vertex (u_3,v_3) since the 3^{rd} and 11^{th} rows are the same .Also, the arcs ((u_4,v_2),(u_1,v_2))and ((u_4,v_2),(u_3,v_2)) can be mapped to the arcs ((u_2,v_2),(u_1,v_2)) and((u_2,v_2),(u_3,v_2)) ,respectively, since the $6^{th}$and 8^{th} rows are the same.Finally the arcs ((u_3,v_1),(u_3,v_2)) and ((u_3,v_1),(u_3,v_4)) can be mapped to the arcs ((u_3,v_3),(u_3,v_2)) and ((u_3,v_3),(u_3,v_4)) ,respectively, since the 3^{th} and 11^{th} rows are the same.And so on, see Figure 6.

VI. THE COMPOSITION OF DIGRAPHS

A. Definition

The composition $D_1[D_2]$of two simple diagraphs is a simple diagraphs with V($D_1[D_2]$)=V_1xV_2. The vertices u=(u_1,u_2) and v=(v_1,v_2) are adjacent if either u_1 is adjacent to $v_1$and u_2=$v_2$or u_1 = v_1 and u_2 is adjacent to v_2.

B. Definition

Let D_1,D_2,D_3 and D_4 be simple diagraphs.Let f: D_1 → D_3 and g:D_2 → D_4be diagraph maps.By the composition dimap f[g]: $D_1[D_2]$ → $D_3[D_4]$ we mean a map defined as follows

(i)Ifv=(v_1,v_2)ϵV($D_1[D_2]$)=V_1xV_2,then f[g]{(v_1,v_2)}={(f(v_1),g(v_2))} ϵV($D_3[D_4]$)

(ii)Let e={({v_1}$_i$,{v_2}$_j$),({v_1}$_k$,{v_2}$_l$)}.If {v_1}$_i$={v_1}$_k$ and {v_2}$_j$ is adjacent to {v_2}$_l$,then f[g]{e}={({v_1}$_i$,g{v_2}$_j$),({v_1}$_i$,g{v_2}$_l$)}.Also, if {v_2}$_j$={v_2}$_l$ and {v_1}$_i$ is adjacent to {v_1}$_k$, thenf[g]{e}={(f{v_1}$_i$,{v_2}$_j$),(f{v_1}$_k$,{v_2}$_j$)}.

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THEOREM

Let D_1, D_2, D_3 and D_4 be digraphs. letf:D_1D_3 and g :D_2D_4be digraph maps.Then the composition dimap f[g]ϵ Đ($D_1[D_2]$, $D_3[D_4]$) is a digraph folding if f ϵ Đ(D_1,D_3) and g ϵ Đ(D_2,D_4) are digraph foldings .

Proof:Let f and g be digraphfolding, then

(i)Foreach vertex v=(v_1,v_2)ϵV($D_1[D_2]$)=V_1xV_2, f[g]{(v_1,v_2)}={(f(v_1),g(v_2))}.But f(v_1)ϵ V(D_3) and g(v_2) ϵ V(D_4) ,then {(f(v_1),g(v_2))} ϵ V($D_3[D_4]$, i.e., f[g] maps vertices to vertices.

(ii)Let e={({v_1}$_i$,{v_2}$_j$),({v_1}$_k$,{v_2}$_l$)} and suppose {v_1}$_i$is adjacent to {v_1}$_k$, then there exists an arc {({v_1}$_i$,{v_1}$_k$)} ϵ A_1, since f is a digraph folding and {({v_1}$_i$,{v_1}$_k$)} ϵ A_1, then f[g]{e} ϵ A($D_3[D_4]$).Now, if {v_1}$_i$={v_1}$_k$and {v_2}$_j$is adjacent to {v_2}$_l$,thenf[g]{e}={({v_1}$_i$,g{ v_2}$_j$),({v_1}$_i$,g{v_2}$_l$)}, since {v_2}$_j$is adjacentto {v_2}$_l$, thenthere exists an arc{({v_2}$_j$,{v_2}$_l$)} ϵ A_2 such that {(g{v_2}$_j$,g{v_2}$_l$)} ϵ A_3 ,i.e., g{v_2}$_j$ $\neq$ g{v_2}$_l$ andhencef[g]{e} ϵ A($D_3[D_4]$), i.e., f[g] maps arcs to arcs.The converse is not true since if f[g] is a digraph folding and for g,is not a digraphfolding.In this case for g,maps an arc to a vertex, say f(u_1,v_1)= (u_3,u_3), $u_3$$\epsilon$ V(D_3).

Then f[g]{(u_1,{v_2}$_i$),(v_1,{v_2}$_j$)}={(f(u_1),g{v_2}$_i$),(f(v_1),g{v_2}$_j$)}={(u_3, g{v_2}$_i$),(u_3, g{v_2}$_j$)} which is an arc of $D_3[D_4]$.

A. Example

Let D_1, D_2,f and g be the digraphs and digraph foldings given in Example (A). The adjacency matrix of $D_1[D_2]$ is as follows:

$M(D_1[D_2])$ =

	(u_1,v_1)	(u_2,v_1)	(u_3,v_1)	(u_4,v_1)	(u_1,v_2)	(u_2,v_2)	(u_3,v_2)	(u_4,v_2)	(u_1,v_3)	(u_2,v_3)	(u_3,v_3)	(u_4,v_3)
(u_1,v_1)	0	1	0	1	1	1	0	1	0	1	0	1
(u_2,v_1)	0	0	1	0	0	0	1	1	0	0	1	0
(u_3,v_1)	0	0	0	0	0	0	1	0	0	0	0	0
(u_4,v_1)	0	0	1	0	0	0	1	1	0	0	1	0
(u_1,v_2)	0	1	0	1	0	1	0	1	0	1	0	1
(u_2,v_2)	0	0	1	0	0	0	1	0	0	0	1	0
(u_3,v_2)	0	0	0	0	0	0	0	0	0	0	0	0
(u_4,v_2)	0	0	1	0	0	0	1	0	0	0	1	0
(u_1,v_3)	0	1	0	1	1	1	0	1	0	1	0	1
(u_2,v_3)	0	0	1	0	0	1	1	0	0	0	1	0
(u_3,v_3)	0	0	0	0	0	0	1	0	0	0	0	0
(u_4,v_3)	0	0	1	0	0	0	1	1	0	0	1	0

Nowa digraph folding f[g]: $D_1[D_2]$$D_1[D_2]$can be defined as follows:f[g]{(u_4,v_1),(u_4,v_2),(u_4,v_3),(u_1,v_1),(u_2,v_1),(u_3,v_1)}={(u_2,v_1),(u_2,v_2), (u_2,v_3),(u_1,v_3),(u_2,v_3),(u_3,v_3)}. Also, f[g]{((u_4,v_1),(u_3,v_1)),((u_3,v_2),(u_3,v_1)),((u_2,v_2),(u_3,v_1))}={((u_2,v_1),(u_3,v_3)),((u_3,v_2),(u_3,v_3)), ((u_2,v_2),(u_3,v_3))}, and so on, see Figure 6.

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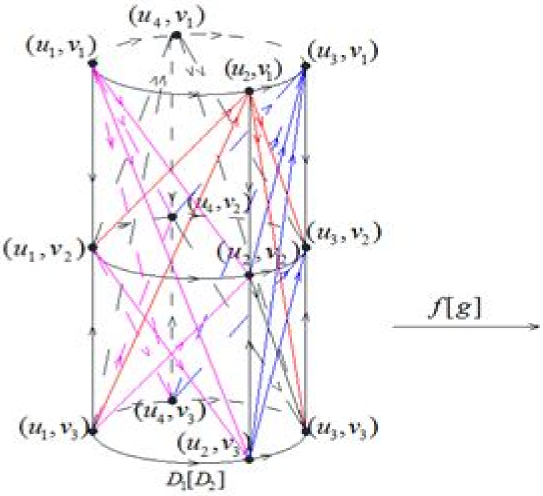

Figure 15: $f[g]\{(u_4,v_1),(u_4,v_2),(u_4,v_3),(u_1,v_1),(u_2,v_1),(u_3,v_1)\}=\{(u_2,v_1),(u_2,v_2), (u_2,v_3),(u_1,v_3),(u_2,v_3),(u_3,v_3)\}$

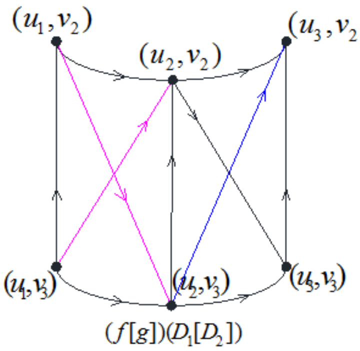

Figure 16: $f[g]\{(u_4,v_1),(u_4,v_2),(u_4,v_3),(u_1,v_1),(u_2,v_1),(u_3,v_1)\}=\{(u_2,v_1),(u_2,v_2), (u_2,v_3),(u_1,v_3),(u_2,v_3),(u_3,v_3)\}$

We can describe the digraph foldings by using $M(D_1)$,$M(D_2)$ and $M(D_1 [D_2])$.For example, from $M(D_1 [D_2])$ we can see that the vertex (u_4,v_1) can be mapped to the vertex (u_2,v_1) since the second and fourth rows have the same entries. Also, the arc $((u_1,v_1),(u_4,v_1))$ can be mapped to the arc $((u_1,v_1),(u_2,v_1))$ since the second and fourth rows are the same. Also the vertex (u_1,v_1) can be mapped to the vertex (u_1,v_3) since 1st and 9^{th} rows have the same entries, and so on.

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