Integral solutions of Ternary Cubic Diophantine equation

\[8\alpha^2 - 5\beta^2 = 3\gamma^3\]

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ABSTRACT: The ternary cubic Diophantine equation given by \[8\alpha^2 - 5\beta^2 = 3\gamma^3\] is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

KEYWORDS: Ternary cubic, integral solutions, polygonal numbers.

I. INTRODUCTION

Ternary quadratic equations are rich in variety [1-3]. For an extensive review of sizable literature and various problems, one may refer [4-18]. In this communication, we consider yet another interesting ternary cubic equation \[8\alpha^2 - 5\beta^2 = 3\gamma^2\] and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions and special polygonal numbers are presented.

II. NOTATIONS USED

- \(t_{m,n}\) - Polygonal number of rank ‘n’ with size ‘m’
- \(CP_{m,n}\) - Centered Pyramidal number of rank ‘n’ with size ‘m’
- \(Pr_n\) - Pronic number of rank ‘n’
- \(P_{n,m}\) - Pyramidal number of rank ‘n’ with size ‘m’
- \(F_{m,n}\) - Figurative number of rank ‘n’ with size ‘m’
- \(Gn_{n,m}\) - Gnomic number of rank ‘n’

III. METHOD OF ANALYSIS

The Cubic Diophantine equation with three unknowns to be solved for its non zero distinct integral solutions is

\[8\alpha^2 - 5\beta^2 = 3\gamma^3\] (1)

We illustrate methods of obtaining non Zero distinct integer solutions to (1)

On substituting the linear transformations

\[
\alpha = X_1 + 5T_1; \quad \beta = X_1 + 8T_1
\]

in (1), leads to

\[
X_1^2 - 40T_1^2 = \gamma^3
\]

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Assume
\[ y_1 = y_1(a, b) = X^2 - 40Y^2; \quad a, b > 0 \] (4)

(3) is solved through different approaches and different patterns of solutions thus obtained for (1) are illustrated below:

**Pattern I**

Equation (3) can be written as
\[ X_1 + \sqrt{40} T_1 = \left( X + \sqrt{40} Y \right) \left( X - \sqrt{40} Y \right) \] (5)

Which is equivalent to the system of equations
\[
\begin{align*}
X_1 + \sqrt{40} T_1 &= (X_1 + \sqrt{40} Y)^3 \\
X_1 - \sqrt{40} T_1 &= (X_1 - \sqrt{40} Y)^3
\end{align*}
\] (6)

Equating rational and irrational parts in (6) we get
\[
\begin{align*}
\alpha &= \alpha(X,Y) = X^3 + 200Y^3 + 15X^2Y + 120XY^2 \\
\beta &= \beta(X,Y) = X^3 + 320Y^3 + 24X^2Y + 120XY^2 \\
\gamma &= \gamma(X,Y) = X^2 - 40Y^2
\end{align*}
\]

**Properties**

1. \( \beta(1,Y) - \alpha(1,Y) - 60 S O_y \equiv 0 (mod\ 23) \)
2. \( \beta(X,1) - \alpha(X,1) - t_{4, x} - 120 \equiv 0 \)
3. \( \alpha(X,1) + \beta(X,1) - SO_x - 39 Pr_x \equiv 520 (mod\ 101) \)
4. \( \alpha(X,1) - CP_x^6 - 15 Pr_x \equiv 200 (mod\ 21) \)
5. \( \gamma(X,Y) - t_{4, x} - 40 \equiv 0 \)
6. Each of the following expression represents a nasty number
   a. \( \beta(0,1) - \alpha(0,1) \)
   b. \( \gamma(1,2) + \gamma(1,1) \)
7. \( \frac{1}{12} \alpha(1,1) \) represents a perfect number.
8. Each of the following expression can be expressed as a difference of two square numbers
   a. \( \alpha(1,1) \)
   b. \( \beta(2,2) \)
   c. \( \gamma(2,2) \)
9. Each of the following expression represents a perfect square
   a. \( \alpha(2,2) + \alpha(1,1) + \alpha(1,0) \)
   b. \( \beta(2,2) + \beta(1,0) \)
   c. \( \beta(0,1) + \beta(1,1) - \beta(1,0) \)
   d. \( \gamma(2,2) - \gamma(0,1) \)
10. Each of the following expression represents a cubical integer
    a. \( \alpha(0,2) - 3\alpha(0,1) \)
    b. \( \alpha(1,2) - \alpha(0,2) + \alpha(1,0) \)
    c. \( \alpha(2,0) + \alpha(1,0) \)
Pattern II

One may write (3) as

\[ X_1^2 - 40 T_1^2 = \gamma^3 \cdot 1 \]  \hspace{1cm} (7)

Write 1 as

\[ 1 = \frac{(\gamma + \sqrt{7}) (\gamma - \sqrt{7})}{9} \]  \hspace{1cm} (8)

Using (4), (5) and (8) in (7) and applying the method of factorization and equating positive factors, we get

\[ X_1 + \sqrt{40} T_1 = \frac{1}{3} \left( 7 + \sqrt{40} \right) (X + \sqrt{40} Y)^3 \]  \hspace{1cm} (9)

Equating rational and irrational parts of (9), we have

\[ X_1 = \frac{1}{3} (7X^3 + 1600 Y^3 + 120 X^2 Y + 840 XY^2) \]

\[ T_1 = \frac{1}{3} (X^3 + 280 Y^3 + 21 X^2 Y + 120 XY^2) \]

Employing (2), the values of X and Y satisfying (1) are given by

\[
\begin{aligned}
\alpha &= \alpha(X,Y) = 4X^3 + 1000Y^3 + 75X^2Y + 480XY^2 \\
\beta &= \beta(X,Y) = 5X^3 + 1280Y^3 + 96X^2Y + 600XY^2 \\
\gamma &= \gamma(X,Y) = X^2 - 40Y^2
\end{aligned}
\]

Properties

1. \( 4\beta(X,1) - 5\alpha(X,1) - t_{4,3X} - 120 \equiv 0 \)
2. \( \alpha(X,1) + \beta(X,1) - 18 P_5^x - 162 P_5^x \equiv 2280 \ (\text{mod } 918) \)
3. \( 4\beta(1,y) - 5\alpha(1,y) - 120CP_5^y \equiv 0 \ (\text{mod } 9) \)
4. \( \beta(x,1) - \alpha(x,1) - 2P_5^x + 20P_5^x \equiv 280 \ (\text{mod } 100) \)
5. \( 4\beta(1,y) - 5\alpha(1,y) - 605O_5^y \equiv 0 \ (\text{mod } 23) \)
6. Each of the following expression represents a cubical integer
   a. \( \gamma(2,2) - \alpha(2,0) + \gamma(1,0) \)
   b. \( \alpha(3,0) - \alpha(1,0) - \beta(1,0) - \gamma(1,0) \)
   c. \( \gamma(3,3) - \gamma(1,3) \)
7. Each of the following expression represents a perfect number
   a. \( \beta(3,0) - \alpha(3,0) + \gamma(1,0) \)
   b. \( \beta(1,0) + \gamma(1,0) \)
8. Each of the following expression represents a Nasty number
   a. \( \frac{1}{2} \alpha(3,0) \)
   b. \( \gamma(1,1) - \gamma(1,2) \)
   c. \( \alpha(3,0) - 3\alpha(1,0) \)
   d. \( \frac{1}{2} \gamma(0,3) \)
   e. \( \alpha(3,0) + 3\alpha(1,0) \)
   f. \( \beta(3,0) - 3\beta(1,0) \)
9. \( \gamma(3,3) \) can be expressed as a sum of cube numbers.
10. Each of the following can be expressed as a perfect squares
   a. \( \alpha(1.0) + \beta(1.0) - \gamma(0.2) \)
   b. \( \beta(1.1) + \gamma(0.1) - \beta(1.0) \)
   c. \( \beta(1.1) - \beta(0.2) - \beta(1.0) \)
   d. \( \alpha(1.1) + \beta(0.2) + \gamma(1.0) \)
   e. \( \alpha(1.1) - \gamma(0.1) + \gamma(1.0) \)

Pattern III
One may write (3) as
\[
X_1^2 - 40T_1^2 = \gamma^3 \cdot 1
\] (10)

Write 1 as
\[
1 = \frac{(11+\sqrt{40})(11-\sqrt{40})}{\sqrt{40}}
\] (11)

Using (4), (5) and (11) in (10) and applying the method of factorization and equating positive factors, we get
\[
X_1 + \sqrt{40}T_1 = \frac{1}{9} (11 + \sqrt{40})(X + \sqrt{40})Y^3
\] (12)

Equating rational and irrational parts of (12), we have
\[
X_1 = \frac{1}{9}(11X^3 + 1600Y^3 + 120X^2Y + 1320XY^2)
\]
\[
T_1 = \frac{1}{9}(X^3 + 440Y^3 + 33X^2Y + 120XY^2)
\]

As our aim is to find integer solutions choosing \( X=3x \), \( Y=3y \), we obtain as follows
\[
\begin{align*}
\alpha &= \alpha(X,Y) = \frac{1}{9}(16X^3 + 3800Y^3 + 385X^2Y + 1920XY^2) \\
\beta &= \beta(X,Y) = \frac{1}{9}(19X^3 + 5120Y^3 + 384X^2Y + 2280XY^2) \\
\gamma &= \gamma(X,Y) = X^2 - 40Y^2
\end{align*}
\]

employing (2), the values of \( X \) and \( Y \) satisfying (1) are given by
\[
\begin{align*}
\alpha &= \alpha(x,y) = 48x^3 + 11400x^3 + 855x^2y + 5760xy^2 \\
\beta &= \beta(x,y) = 57x^3 + 15360y^3 + 1152x^2y + 68440xy^2 \\
\gamma &= \gamma(x,y) = 9x^2 - 360y^2
\end{align*}
\]

Properties
1. \( \beta(x,1) - \alpha(x,1) - 9CP_x^6 - 297Pr_x - Gx_0 \equiv 3960 \mod 11 \)
2. \( \beta(1,y) - \alpha(1,y) - 3960CP_x^6 - 297Pr_x - 783t_4y - 9 \equiv 0 \)
3. \( \beta(x,1) - \alpha(x,1) - 18P_x^6 - 288Pr_x \equiv 3960 \mod 792 \)
4. \( \beta(x,1) - 114 P_x^6 - 1095Pr_x \equiv 15360 \mod 5745 \)
5. \( \alpha(2,y) - 22800 P_x^6 - 120Pr_x \equiv 384 \mod 3300 \)
6. Each of the following expression represents a perfect square
Observation on the Ternary Cubic Equation

\[ \alpha(1.1) + y(1.2) + y(1.0) \]

7. \( \beta(2.2) - y(3.3) \) represents a cubic number
8. \( y(3.3) \) can be expressed as a difference of two square numbers.

IV. CONCLUSION

In this paper, we have presented three different patterns of non-zero distinct integer solutions of ternary cubic Diophantine equation \( 8\alpha^2 = 5\beta^2 = 3y^3 \) and relations between solutions and special numbers are also obtained. To conclude, one may search for other patterns of solutions and their corresponding properties.

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