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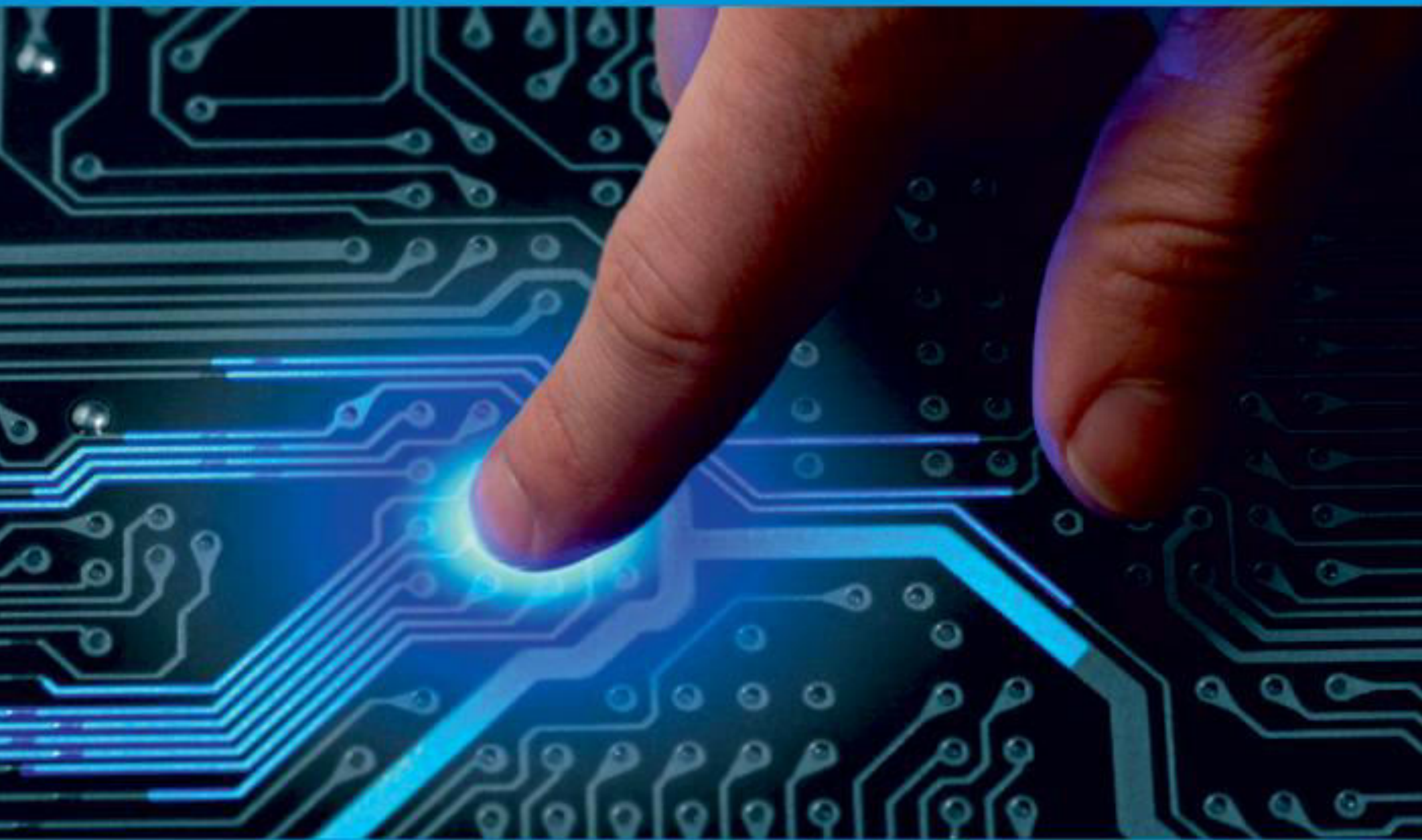

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Some Results Involving Multiparameter Generating Functions Associated with Hurwitz-Lerch Zeta function

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ABSTRACT: In this paper two results for generating functions associated with multiparameter Hurwitz Lerch Zeta function are established. These generating functions are studied and established by Srivastava. A few known and new integrals are also, obtained as special cases of our main results in this paper.

KEYWORDS: Zeta function, Lerch-Zeta function, Riemann Zeta function, Generalized hyper geometric function, Wrights generalized hyper geometric function, Z Hurwitz- Lerch Zeta function.

I. INTRODUCTION AND PRILIMINARIES:

The Riemann Zeta function $\varsigma(s)$, the Hurwitz Zeta function $\varsigma(s,a)$ and the Lerch Zeta function $l_s(\xi)$ defined as see for detail [1,Chp.1]

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\phi(1,s,1)=\zeta(s,1)\left(\operatorname{Re}(s)>1\right) \tag{1.1}$$

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}=\phi(1,s,a)\left(\operatorname{Re}(s)>1;a\in C\setminus Z_0^-\right), \tag{1.2}$$

$$l_s(\xi)=\sum_{n=0}^{\infty}\frac{e^{2n\pi i\xi}}{(n+1)^s}=\phi\left(e^{2\pi i\xi},s,1\right)\left(\operatorname{Re}(s)>1;\xi\in R\right) \tag{1.3}$$

The Hurwitz-Lerch Zeta function is a generalization of the Riemann Zeta function. It is defined as Srivastava [5]

$$\phi(z,s,a)=\sum_{l=0}^{\infty}\frac{z^l}{(l+a)^s} \tag{1.4}$$

($a\in C/Z_0^-;s\in C$ when |z|<1; $R(s)>1$ when |z|=1)

The integral representation of above defined Hurwitz Lerch zeta function is given by Erdelyi et al [1, pp. 27, eq 1.11(3)].

$$\phi(z,s,a)=\frac{1}{\Gamma s}\int_0^{\infty}\frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt \tag{1.5}$$

$\operatorname{Re}(a)$ > 0, Re(s) >0 when $|z|\le 1(z\ne 1)$, Re(s) >1 when z = 1 at a =0

A generalization of the Hurwitz-Lerch Zeta function is also studied by Goyal and Laddha [10] as follows

$$\phi_\mu^*(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n z^n}{n!(n+a)^s}$$

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Where $\mathrm{Re}(\mu)>0$ and $(\mu)_n$ is the Pochhammer symbol with relation $(\mu)_n=\frac{\Gamma(\mu+n)}{\Gamma(\mu)}$

its integral representation

$$\phi_\mu^*(z,s,a)=\frac{1}{\Gamma s}\int_0^\infty t^{s-1}e^{-at}\left(1-ze^{-t}\right)^{-\mu}dt,$$
$$\min\{R(a),R(s)\}>0;|z|<1 \tag{1.6}$$

The Fox and Wright studied a function ${}_p\psi_q^*$ which is the generalization of the familiar hypergeometric function ${}_pF_q$ and known as wright's generalized hypergeometric function.It is defined as

$${}_p\psi_q^*\left[\begin{matrix}(a_1,A_1),\ldots\ldots,(a_p,A_p);\\(b_1,B_1),\ldots\ldots\ldots,(b_q,B_q);\end{matrix}z\right]=\sum_{n=0}^\infty\frac{(a_1)_{A_1n}\ldots..(a_p)_{A_pn}z^n}{(b_1)_{B_1n}\ldots..(b_q)_{B_pn}n!} \tag{1.7}$$

At $A_i=1(i=1,\ldots..,p)$, $B_j=1(j=1,\ldots..,q)$ it reduces to generalized hypergeometric function ${}_pF_q$.

Further generalization of the above defined Hurwitz-Lerch Zeta function $\phi_\mu(z,s,a)$ and $\phi_\mu^*(z,s,a)$ is recently studied in the following form by Garg et al [7];

$$\phi_{\lambda,\mu,\gamma}(z,s,a)=\sum_{n=0}^\infty\frac{(\lambda)_n(\mu)_n\quad z^n}{(\gamma)_n\,n!(n+a)^s} \tag{1.8}$$

and $$\phi_{\lambda,\mu,\gamma}(z,s,a)=\frac{1}{\Gamma s}\int_0^\infty t^{s-1}e^{-at}{}_2F_1\left[\begin{matrix}\lambda,\mu\\\gamma\end{matrix};ze^{-t}\right]dt \tag{1.9}$$

where $\lambda,\mu\in C,a\in C/Z_0^-,s\in C$ when $|z|<1$ $\mathrm{Re}(s+\upsilon-\lambda-\mu)$ when $|z|=1$

Lin and Srivastava [9] also extended the Hurwitz-Lerch zeta function in the following form.

$$\phi_{\mu,\gamma}^{\rho,\sigma}(z,s,a)=\sum_{n=0}^\infty\frac{(\mu)_{\rho n}z^n}{(\gamma)_{\sigma n}(n+a)^s}$$
$$=\frac{1}{\Gamma s}\int_0^\infty t^{s-1}e^{-at}{}_2\psi_1^*\left[\begin{matrix}(\mu,\rho)\;(1,1)\\(\gamma,\sigma)\end{matrix};ze^{-t}\right]dt \tag{1.10}$$

$\mu\in C;a,\lambda\in C|Z_0^-;\ \rho,\sigma\in R^+;\rho<\sigma$ when $s,z\in C;\rho=\sigma$

and $s\in C$ when $|z|<\delta=\rho^{-p}\sigma^\sigma;\ \rho=\sigma$ and $\mathrm{Re}(s-\mu+v)>1$ when $|z|=\delta$

Bin-Saad [8] established the following generating function for the Hurwitz – Lerch zeta function defined in (1.1)

$$\sum_{n=0}^\infty\frac{(\lambda)_n}{n!}\phi(z,s+n,a)t^n=\sum_{n=0}^\infty\frac{z^n}{(n+a)^{s-\lambda}(n+a-t)^\lambda}=V_\lambda(z;t,s,a) \tag{1.11}$$
$$|t|<|a|$$

When $t\to t/\lambda$ and $|\lambda|\to\infty$, (1.8) becomes

$$\sum_{n=0}^\infty\phi(z,s+n,a)\frac{t^n}{n!}=\sum_{n=0}^\infty\frac{z^n}{(n+a)^s}\exp\left(\frac{t}{n+a}\right)=\psi(z,t,s,a),|t|<\infty \tag{1.12}$$

The extension of Hurwitz Lerch zeta function in multiparameter is defined by Srivastava [5] as follows.

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$$\phi\begin{pmatrix}\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q\\ \lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q\end{pmatrix}(z,s,a)=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\bullet\frac{z^n}{(n+a)^s} \qquad (1.13)$$

$$p,q\in N_0;\lambda_j\in C(j=1,\ldots,p)\;;\quad a,\mu_j\in c/z_0^-(j=1,\ldots,q)$$

$$\rho_j,\sigma_k\in R^+(j=1,\ldots,p,k=1,\ldots,q)$$

They introduced the following generating relations associated with multiparameter Hurwitz- Lerch zeta function defined in (1.10)

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{(\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q)}(z,s+n,a)t^n$$

$$=\sum_{k=0}^{\infty}\frac{E_k z^k}{(k+a)^{s-\lambda}(k+a-t)^{\lambda}}=\Omega_\lambda(z,t;s,a) \qquad |t|<|a| \qquad (1.14)$$

where

$$E_k=\frac{\prod_{j=1}^{p}(\lambda_j)_{k\rho_j}}{k!\prod_{j=1}^{q}(\mu_j)_{k\sigma_j}}$$

$$k\in N_0,$$

When $t\to\frac{t}{\lambda}$ and $|\lambda|\to\infty$ the generating (1.11) yields

$$\sum_{n=0}^{\infty}\phi\begin{pmatrix}\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q\\ \lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q\end{pmatrix}(z,s+n,a)\frac{t^n}{n!}=\sum_{k=0}^{\infty}\frac{E_k z^k}{(k+a)^s}\exp\left(\frac{t}{k+a}\right)$$

$$=\theta(z,t;s,a) \qquad (|t|<\infty) \qquad (1.15)$$

The truncated form of the generating function $\Omega_\lambda(z,t;s,a)$ and $\theta(z,t;s,a)$ are also defined by Srivastava [6] respectively.

$$\Omega_\lambda^{0,r}(z,t;s,a)=\sum_{k=0}^{r}\frac{E_k z^k}{(k+a)^{s-\lambda}(k+a-t)^{\lambda}} \quad ,\ r\in N_0$$

$$\Omega_\lambda^{r+1,\infty}(z,t;s,a)=\sum_{r+1}^{\infty}\frac{E_k z^k}{(k+a)^{s-\lambda}(k+a-t)^{\lambda}} \quad ,\ r\in N_0$$

$$\theta^{0,r}(z,t;s,a)=\sum_{k=0}^{r}\frac{E_k z^k}{(k+a)^s}\exp\left(\frac{t}{k+a}\right) \quad ,\ r\in N_0$$

and

$$\theta^{r+1,\infty}(z,t;s,a)=\sum_{k=r+1}^{\infty}\frac{E_k z^k}{(k+a)^s}\exp\left(\frac{t}{k+a}\right) \quad ,\ r\in N_0$$

Which satisfy the following decomposition formulas :

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$$\Omega_{\lambda}^{(0,r)}(z,t;s,a)+\Omega_{\lambda}^{(r+1,\infty)}(z,t;s,a)=\Omega_{\lambda}(z,t;s,a)$$
$$\theta^{(0,r)}(z,t;s,a)+\theta^{(r+1,\infty)}(z,t;s,a)=\theta(z,t;s,a) \tag{1.16}$$

The integral representation formula for these generating functions are defined as follows (Srivastava [5], [6]):

$$\Omega_{\lambda}(z,\omega;s,a)=\frac{1}{\Gamma s}\int_0^{\infty} t^{s-1}e^{-at}\ {}_p\psi_q^{*}\left[\begin{matrix}(\lambda_1,\rho_1),\ldots\ldots,(\lambda_p,\rho_p)\\(\mu_1,\sigma_1),\ldots\ldots,(\mu_q,\sigma_q)\end{matrix};ze^{-t}\right]{}_1F_1(\lambda;s;\omega t)dt$$

and

$$\theta(z,\omega;s,a)=\frac{1}{\Gamma s}\int_0^{\infty} t^{s-1}e^{-at}\ {}_p\psi_q^{*}\left[\begin{matrix}(\lambda_1,\rho_1),\ldots(\lambda_p,\rho_p)\\(\mu_1,\sigma_1),\ldots(\mu_q,\sigma_q)\end{matrix};ze^{-t}\right]{}_0F_1(-;s;\omega t)dt$$

where $\{\min R(a),R(s)\}>0$

Results Required

The following result is also required here [3, pp 181-184] :

For $f(t)=(\eta-\xi)+\rho(t-\xi)+\sigma(\eta-t)$ we have ,

$$\int_{\xi}^{\eta}(t-\xi)^{\alpha-1}(\eta-t)^{\beta-1}\left[f(t)\right]^{-\alpha-\beta}dt=B(\alpha,\beta)(\eta-\xi)^{-1}(1+\rho)^{-\alpha}(1+\sigma)^{-\beta} \tag{1.17}$$

$$\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0;(\eta-\xi)+\rho(t-\xi)+\sigma(\eta-t)\neq 0,\xi\neq\eta$$

We have established the following results involving functions related to Hurwtiz-Lerch Zeta functions.

2.Main Results

For $f(t)=(\eta-\xi)+\rho(t-\xi)+\sigma(\eta-t)$, $h(t)=(t-\xi)^{\gamma}\left[f(t)\right]^{-\gamma}$, $v(t)=(t-\xi)^{\gamma}(\eta-t)^{\gamma}\left[f(t)\right]^{-2\gamma}$

and for the sequence of coefficient $\{E_k\};k\in N_0$ the following main results are established here.

Result-1

$$\int_{\xi}^{\eta}(t-\xi)^{\alpha-1}(\eta-t)^{\beta-1}\left[f(t)\right]^{-(\alpha+\beta)}\Omega_{\lambda}\left(z,\omega(t-\xi)^{\gamma}(\eta-t)^{\delta}\left[f(t)\right]^{-(\gamma+\delta)},s,a\right)dt$$

$$=B(\alpha,\beta)(\eta-\xi)^{-1}(1+\rho)^{-\alpha}(1+\sigma)^{-\beta}\sum_{k=0}^{\alpha}\frac{E_k z^k}{(k+a)^s}$$
$${}_3\psi_1^{*}\left[\begin{matrix}(\lambda,1),(\alpha,\gamma),(\beta,\delta);\\(\alpha+\beta,\gamma+\delta);\end{matrix}\frac{\omega(1+\rho)^{-\gamma}(1+\sigma)^{-\delta}}{(k+a)}\right]$$

where $n\neq\xi$ $\min\{R(\alpha),R(\beta)\}>0;\ \gamma>0$

(2.1)

Result-2

$$\int_{\xi}^{\eta}(t-\xi)^{\alpha-1}(\eta-t)^{\beta-1}\left[f(t)\right]^{-(\alpha+\beta)}\theta\left(z,\omega(t-\xi)^{\gamma}(\eta-t)^{\delta}\left[f(t)\right]^{-(\gamma+\delta)},s,a\right)dt$$

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$$=(1+\rho)^{-\alpha}(1+\sigma)^{-\beta}(\eta-\xi)^{-1}B(\alpha,\beta)\sum_{k=0}^{\infty}\frac{E_k z^k}{(k+a)^s}\,{}_2\psi_1^*\left[\begin{matrix}(\alpha,\gamma),(\beta,\delta);\ \dfrac{\omega(1+\rho)^{-\gamma}(1+\sigma)^{-\delta}}{(k+a)}\\(\alpha+\beta,\gamma+\delta);\end{matrix}\right]$$

where

$$\eta\neq\xi,\min\{R(\mu),R(\upsilon)\}>0;\gamma>0 \quad (2.2)$$

OUTLINES OF PROOFS

Proof of (2.1):

To prove the result in (2.1), first we denote the LHS of (2.1) by I_1 i.e.

$$I_1=\int_{\xi}^{\eta}(t-\xi)^{\alpha-1}(\eta-t)^{\beta-1}\left[f(t)\right]^{-(\alpha+\beta)}\,\Omega_\lambda\left(z,\omega(t-\xi)^{\gamma}(\eta-t)^{\delta}\left[f(t)\right]^{-(\gamma+\delta)},s,a\right)dt$$

Now on using the definition of Ω_λ given (1.14) and using binomial expansion theorem and then on changing the order of integration and summation we have

$$I_1=\sum_{n,k=0}^{\infty}\frac{E_k z^k}{(k+a)^s}\frac{(\lambda)_n}{n!}\omega^n\int_{\xi}^{\eta}(t-\xi)^{\alpha+\gamma n-1}(\eta-t)^{\beta+\delta n-1}\left[f(t)\right]^{-[(\alpha+\beta)+\eta(\gamma+\delta)]}dt$$

Now evaluating the inner integral with help of (1.17) and on interpreting then n-series in view of (1.7) we atonce arrive at the desired result in (2.1).

The result in (2.2) is proved following the similar lines as to prove the result (2.1) in view of definition of $\theta(z,t;s,a)$ given in (1.15) and using exponential series .

Particular cases:

1.If in results (2.1) and (2.2), we take $\rho=\sigma=0$ and ω is replaced by $\omega(\eta-\xi)^{\gamma+\delta}$ then these results reduce to the known results due to Srivastava[5,eqs. (3.21) and (3.22)] respectively.

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